Mathematics Competition

\$25 prize for the best solution for each of 5 problems.
\$100 prize for solving the most problems throughout the semester.

Problem #5 of five - April 5 to April 19, 2013

Show that for any positive integer n, at least one of n or $n+1$ can be represented in the form $k + S(k)$ for some positive integer k, where $S(k)$ is the sum of the digits of k. For example, $249 = 237 + (2 + 3 + 7)$.

Direct any questions to Kamlesh Parwani, OM 3351, or Keith Wolcott, OM 3341

No complete solutions were submitted to this weeks challenge.

Solution. We use induction on n. For $n = 1$, $n + 1 = 2 = 1 + S(1) = 1 + 1$. Now assume the result for all positive integers less than $n + 1$ and we will show that it is true for $n + 1$. Since the result is true for n we know that either n or $n + 1$ can be represented as $k + S(k)$ for some k. If $n+1$ can, then we are done. If $n = k+S(k)$ for some k then consider $k+1+S(k+1)$. If the units digit of k is not a nine, then $S(k+1) = S(k)+1$ and then $k+1+S(k+1) = k+1+S(k)+1 = n+2$ and we are done. Now we just need to handle the case where the units digit of k is a nine. Suppose in this case that there are m consecutive nines ending with the units digit. Then $k + 1 = 10^m a$ for some integer a and $S(k + 1) = S(k) + 1 - 9m$. Also note that since $k + 1$ has m zeros at the end, that if i is any positive integer less than 10^m then $S(k + 1 + i) = S(k + 1) + S(i)$.

Now by induction, either $9m - 1$ or $9m$ can be expressed as $i + S(i)$ for some positive integer i. Then

$$\begin{aligned}
k + 1 + i + S(k + 1 + i) &= k + 1 + i + S(k + 1) + S(i) \\
&= k + 1 + S(k + 1) + (i + S(i)) \\
&= k + 1 + S(k) + 1 - 9m + (i + S(i)) \\
&= (k + S(k)) + 2 - 9m + (i + S(i)) \\
&= n + 2 - 9m + (i + S(i))
\end{aligned}$$

Now $i + S(i)$ equals either $9m - 1$ or $9m$ so the above simplifies to either $n + 1$ or $n + 2$. Thus we have shown that either $n+1$ or $n+2$ can be expressed as $k+1+i+S(k+1+i)$. By induction, the result is true for all positive integers n.