Mathematics Competition

Problem #1 of five - September 13 to September 27, 2013

A number has been obtained by rearranging the digits of another number. For example, 247 and 724.
a) Can the sum of two such numbers equal 999,999?
b) Can the sum of two such numbers equal to 9,999,999?
c) Generally, if the sum consists of all 9's (say n 9's), then how many such pairs of numbers are there?
(Note: The first (the leftmost) digit of an integer is not zero. If an interior zero of the first number moves to the first position on the left of the rearranged number, it disappears!)
Give a reasoned explanation of your answers.

Direct any questions to Gregory Galperin, OM 3361, or Keith Wolcott, OM 3341

Congratulations to our winners Marika Rosenberger, Jerry Bragg, Megan Childers, and Andrew Smith.

Solution. a) Yes, for example 123,678 and 876,321 or 587,412 and 412,587.

b) No. Suppose that we have two such numbers A and B that sum to 9,999,999. The units digits of A and B must sum to 9 or possibly 19 if there is a "carry". But two single digits can sum to at most 18, so there cannot be a "carry". Then the same is true for the 10's digits, and likewise for every digit. Thus the units digits are a pair that sum to 9, the 10's digits are a pair that sum to 9, etc., which means that we have seven pairs of digits whose sum is 9. Since one number is obtained from the other by rearranging the digits, all seven pairs of digits must be included in each number A and B. Thus the seven digits can be paired off in A which would mean that A has an **even** number of digits. Since there are seven digits, and seven is not even, it cannot be done.

c) In part b) we showed that there are no such pairs if n is odd. Now suppose that n is even. For the general case with n digits, this is a difficult combinatorial problem and there isn't a nice solution. We give one of a couple of solutions that we found, in the cases for $n = 2, 4$, and 6. The problem only says that one number is a rearrangement of the other. For example, 854 is a rearrangement of 8504, but 8504 is not a rearrangement of 854. This case is very similar to when the two numbers are rearrangements of each other, which is the case that we will do. Thus, the first digit cannot be a zero or a nine (if the first digit is a nine then the first digit of the other number is a zero and the zero drops off, so the numbers are not rearrangements of each other). Thus there are 8 choices for the first digit (1 - 8) and the second digit is then determined as the complementary digit that sums to 9. The 8 possibilities are 18, 27, 36, 45, 54, 63, 72, and 81. This gives 8 pairs of numbers:

18, 81	54, 45
27, 72	63, 36
36, 63	72, 27
45, 54	81, 18

Since these are not ordered pairs, they are all listed twice and we really only have four pairs of numbers:

18, 81
27, 72
36, 63
45, 54

Thus, to avoid the duplication, we could have just varied the first digit from 1 to 4. We conclude that for $n = 2$ there are 4 pairs of numbers.

For $n = 4$ there are 4 choices for the first digit on the left and then 10 choices for the second digit. The third and fourth digits are then determined but could be in either order. Thus it seems like $4 \cdot 10 \cdot 2 = 80$ ways to get such a number. The number 80 is not quite correct since in the case that the first two digits are the same, the last two digits are also the same and there is only one way to order them rather than 2 ways. To correct this, we count the ways in which the first two digits are distinct and then add on the cases when they are not distinct.

That is, for $n = 4$ we have the number of pairs A, B with $A + B = 9,999$ equals

$$\begin{aligned}
& \text{(4 choices)(9 choices)(ways to order last 2 digits)} && \text{(2 distinct digits)} \\
& + \text{(4 choices)(1 choice)(ways to order last 2 digits)} && \text{(1 distinct digit)} \\
= \; & 4 \cdot 9 \cdot 2 && \text{(2 distinct digits)} \\
& + 4 \cdot 1 \cdot 1 && \text{(1 distinct digit)} \\
= \; & 76
\end{aligned}$$

Similarly, for $n = 6$ we have the number of pairs A, B with $A + B = 999,999$ equals

$$\begin{aligned}
& 4 \cdot 9 \cdot 8 \cdot (3!) && \text{(3 distinct digits)} \\
& + 4 \cdot 9 \cdot 2 \cdot 3 && \text{(2 distinct digits)} \\
& + 4 \cdot 1 \cdot 1 \cdot 1 && \text{(1 distinct digit)} \\
= \; & 1728 + 216 + 4 \\
= \; & 1948
\end{aligned}$$

The same method can be used for $n = 8$ and larger, but it gets more complicated since you have to consider, for example, in the case when there are two distinct digits, that there could be three of a kind and a single, or there could be two pairs. The cases $n = 10$ and $n = 12$ are even more difficult, and we do not consider them here.