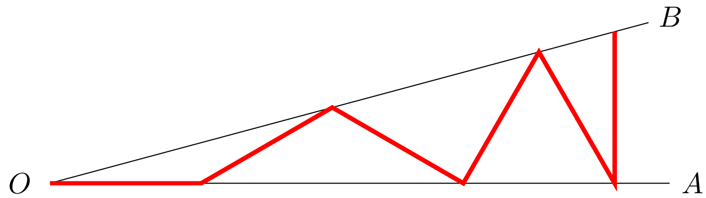
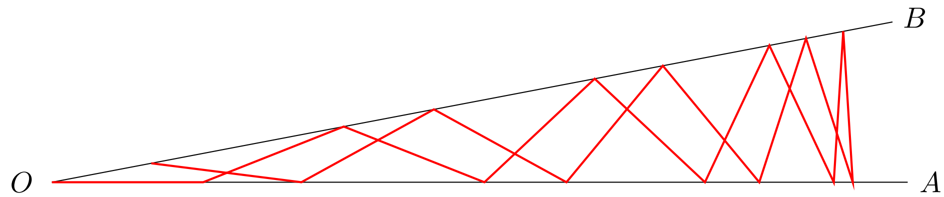
Mathematics Competition

Problem #5 of five - November 15 to December 6, 2013

Angle $\angle AOB$ with measure α is shown below. A grasshopper starts at vertex O, always jumps distance 1, always jumps from one leg of the angle to the other, and never returns the way it just came. If there is no possible jump, the grasshopper stops. The first example below shows a grasshopper that does six jumps and must stop, and the second example shows a case with more jumps that end close to O. Find an angle α that causes the grasshopper to stop at O after 2013 jumps. Explain your solution.





Direct any questions to Gregory Galperin, OM 3361, or Keith Wolcott, OM 3341

Congratulations to our winners, Jerry Bragg and David Stevens.

Solution. First we claim that the angles change as shown in Figure 1. That is, when the grasshopper does its kth jump from point P to point Q, the measure of $\angle APQ$, if P is on leg OA, or the measure of $\angle BPQ$, if P is on leg OB, is $k\alpha$.

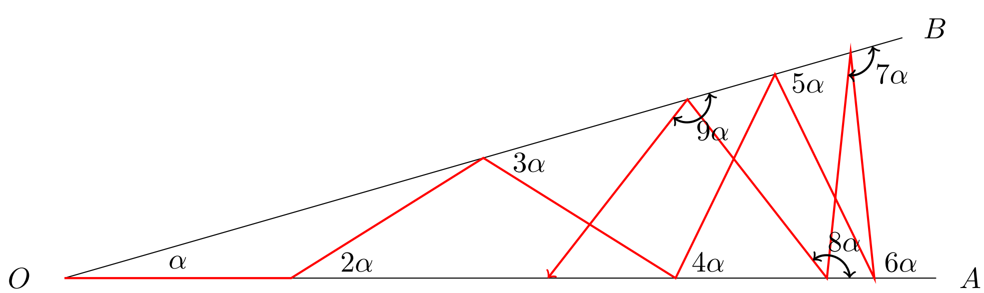


Figure 1: Changing angles.

The proof is by induction on the number of jumps k. When $k = 1$ the jump angle is α. Figure 2 shows the induction step for the proof. The grasshopper has jumped from point P to Q to R. In going from $k\alpha$ to $(k+1)\alpha$ there is always an isosceles triangle (either right side up or right side down) so base angles are equal. Drawing in the line parallel to the bottom leg of the original angle as shown in the figure, and the fact that alternate interior angles are equal, shows that the next angle is $(k+1)\alpha$.

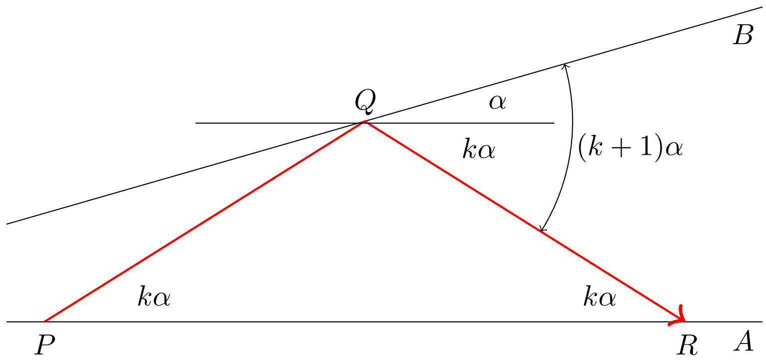


Figure 2: Changing angles when moving away from O.

When the grasshopper is moving closer to the vertex O, the situation is as shown in Figure 3. The grasshopper has jumped from point P to Q to R. As before, angle $k\alpha$ is the measure of $\angle APQ$ and the next angle is the measure of $\angle BQR$. Again, the isosceles triangle has equal base angles and alternate interior angles are equal so we get the same result.

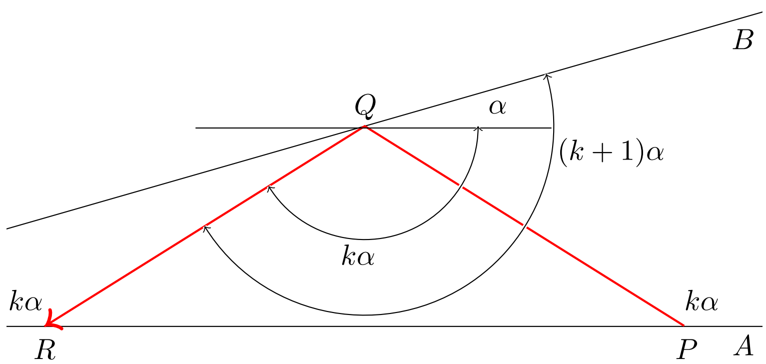


Figure 3: Changing angles when moving toward O.

Thus, we have that the angles increase by α when moving away from O and when moving toward O. When the grasshopper is transitioning from going away to coming back, the only time we don't have isosceles triangles as in Figure 2 or 3, is when the jump angle is 90°. In this case the isosceles triangle is degenerate with P and R the same point, but the result is the same, as shown in Figure 4.

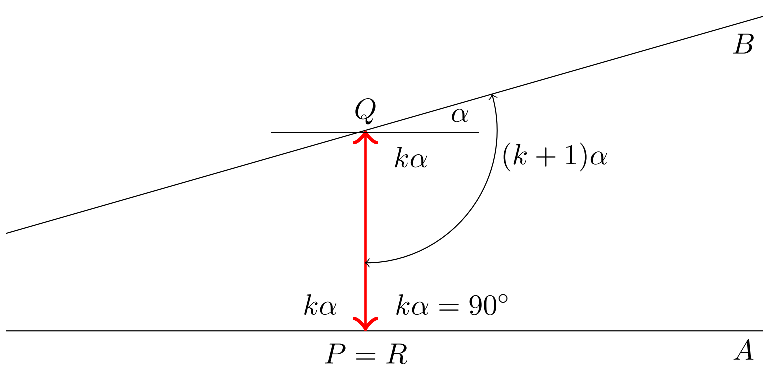


Figure 4: Jump angle $k\alpha = 90°$.

To get back to the vertex O, the last jump must be along one of the legs of the angle, so the angle of its last jump is $180°$. It cannot be along the leg that it started on (since it cannot return along its path) so it must end on the other leg. Thus the path must be symmetric across an axis that is half of the angle α as is shown in Figure 5. Because of the symmetry and the single jump on the far right, the number of jumps must be odd. Since the last jump must be at $180°$, and the 2013th jump must have angle 2013α we have that

$$2013\alpha = 180$$

or

$$\alpha = \frac{180}{2013} \text{ degrees.}$$

Figure 5 shows 15 jumps with $\alpha = \frac{180}{15} = 12$ degrees.

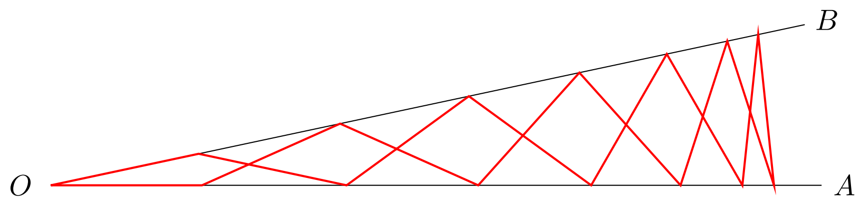


Figure 5: 15 Jumps with $\alpha = 12°$.

This completes the solution to the problem, but there are some interesting additional results that we can deduce.

Remark 1: If the grasshopper reaches its maximum distance from the vertex O and stops, then the last jump was done at a $90°$ angle (thus it stops since it would have to return on the same path). This happens any time the grasshopper does k jumps with $k\alpha = 90$ or $\alpha = \frac{90}{k}$. This is illustrated in the first figure in the statement of the problem with $\alpha = \frac{90}{6} = 15°$.

The maximum distance d, that the grasshopper could possibly get from O is achieved in this case when $\alpha = \frac{90}{k}$, so $\sin\alpha = \frac{1}{d}$ and $d = \frac{1}{\sin\alpha}$. This fact was useful for the figures on these pages which all used 6% larger than d for the lengths of the legs of the angle.

Remark 2: There are essentially three types of paths that the grasshopper can take, depending on the angle α. All three possible cases are finite in length.

- **Type 1:** As in the given problem, the grasshopper stops at the vertex O after $2k+1$ jumps, when $\alpha = \frac{180}{2k+1}$. See Figure 5 for an example.

- **Type 2:** The grasshopper stops at the point farthest from the vertex O that it could possibly reach ($d = \frac{1}{\sin\alpha}$) which is just after making a jump at a $90°$ angle. Thus, $\alpha = \frac{90}{k}$ and the grasshopper makes k jumps. See the first figure in the statement of the problem for an example with $\alpha = \frac{90}{6} = 15°$.

- **Type 3:** The angle $\alpha \neq \frac{180}{2k+1}$ and $\alpha \neq \frac{90}{k}$ for all integers k, and the grasshopper jumps with angles $2\alpha, 3\alpha, 4\alpha, \ldots, k\alpha$ until $180 - \alpha < k\alpha < 180$. Figure 6 shows this situation where the grasshopper must stop. A case of this is illustrated in the second figure of the problem statement when $\alpha = 10.8°$ and there are $k = 16$ jumps.

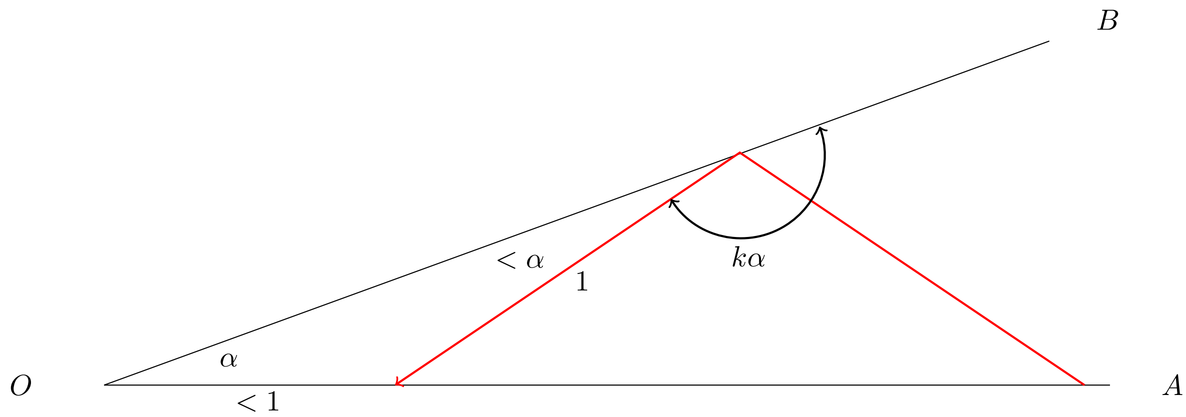


Figure 6: Last jump when returning.

Remark 3: An interesting variation of this problem is when the legs OA and OB are extended through the vertex O and the grasshopper can jump onto the "other side". The same arithmetic sequence of angles continues in this case. See Figure 7 which is type 1 with $\alpha = \frac{180}{15} = 12°$.

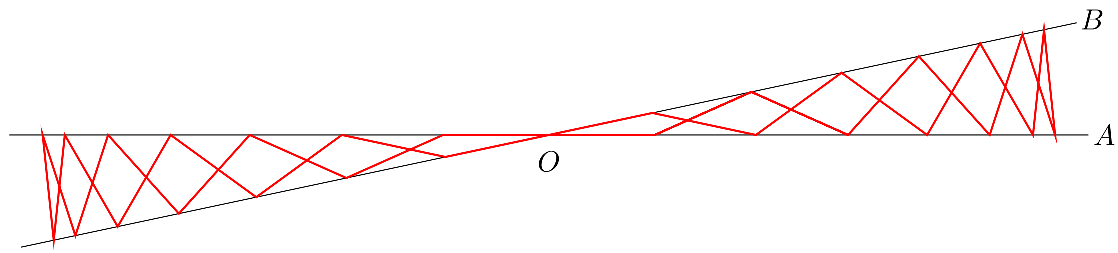


Figure 7: Type 1: 30 Jumps with $\alpha = \frac{180}{15} = 12°$. Then it repeats.

See Figure 8 for an example of Type 2. Technically, the whole path is not covered, since if it starts at the vertex O it would stop at the point near A. But if it started at the point near A, then it would traverse the entire path to the left and then stop.

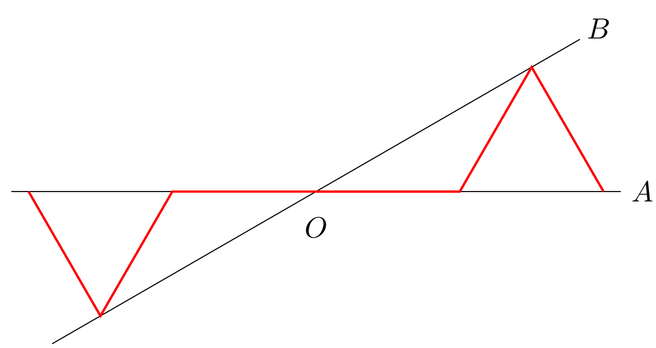


Figure 8: Type 2: $\alpha = \frac{90}{3} = 30°$.

Type 3 again has $\alpha \neq \frac{180}{2k+1}$ and $\alpha \neq \frac{90}{k}$ for all integers k, but this case breaks into 2 cases. In the case that α is irrational, $k\alpha$ will never be a multiple of 90 or 180 so the grasshopper never stops (if the path were to cycle, the first jump would be repeated and we would have $k\alpha =$ a multiple of 180). On the other hand, if α is rational, then some multiple of α is a multiple of 90, so the path repeats eventually.

Figure 9 shows another example with $\alpha = 10.5°$, that repeats much sooner. Since $(60)(10.5) = (7)(90)$ it repeats after 60 jumps.

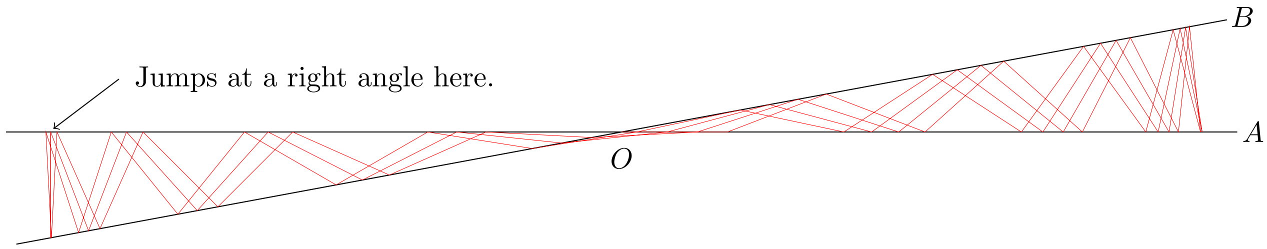


Figure 9: Type 3: 60 jumps with $\alpha = 10.5°$. Then it repeats.

Figure 10 shows another example with $\alpha = 10.4°$.. Since $(225)(10.4) = (26)(90)$ it repeats after 225 jumps.

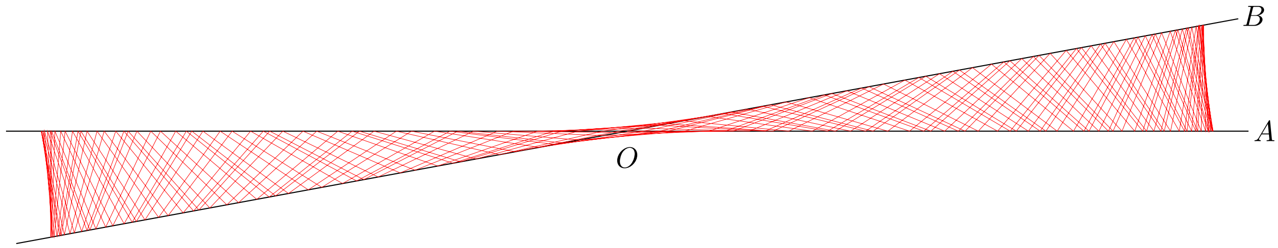


Figure 10: Type 3: 225 jumps with $\alpha = 10.4°$. Then it repeats.

Figure 11, show type 3 with $\alpha = 10.9°$. This angle is rational and since $(900)(10.9) = (109)(90)$ the path repeats after 900 jumps.

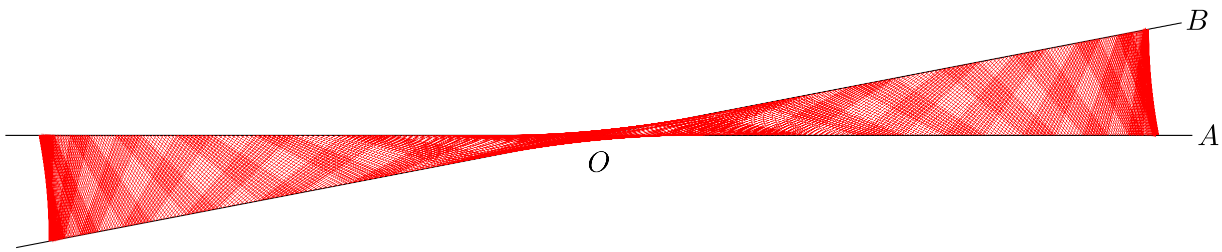


Figure 11: Type 3: 900 jumps with $\alpha = 10.9°$. then it repeats.