Mathematics Competition

Problem #4 of five - November 1 to November 15, 2013

An $n \times n$ table is filled with distinct numbers (not necessarily integers). The smallest number in each row is colored red, and it turns out that all the red numbers are located in n different columns. Then the smallest number in each column is colored blue, and it turns out that all the blue numbers are located in n different rows. Prove that the set of n red numbers equals the set of n blue numbers.

Direct any questions to Gregory Galperin, OM 3361, or Keith Wolcott, OM 3341

Congratulation to our winners, Jerry Bragg and David Stevens.

Solution. We give two different solutions. The first is Jerry Bragg's solution.

Proof 1: Since the numbers of the $n \times n$ table are all distinct, there is a smallest entry a_1. Since a_1 is the smallest entry in the table, it is the smallest entry in its row and in its column. Thus, the red and the blue numbers for that row and column are both equal to a_1. Now delete the row and column that a_1 is in and let a_2 be the smallest number in the table that remains. Again, a_2 must be equal to the red and blue numbers in the row and column that a_2 is in (the red and blue numbers can't have been in the previous row and column that was removed since the red numbers are all in different columns and the blue numbers are all in different rows). Now delete the row and column that contain a_2 and continue in the same way showing that each red number equals a blue number until they are all equal.

Proof 2: Suppose the two sets are different. That means that there is a row with minimum r_1 that is in a column whose minimum b_1 is different from r_1.

Now assuming that r_k and b_k have been defined, define r_{k+1} and b_{k+1} as follows:

Let r_{k+1} be the minimum of the row that b_k is in. Then r_{k+1} is different from b_k (if the same, then two red numbers, r_k and r_{k+1}, are in the same column and we are given that they are all in different columns).

Let b_{k+1} be the minimum of the column that r_{k+1} is in. Then b_{k+1} is different from r_{k+1} (if the same, then two blue numbers, b_k and b_{k+1}, are in the same row and we are given that they are all in different rows).

This defines a sequence $r_1, b_1, r_2, b_2, r_3, b_3, \ldots$. Since each r_i and b_i is the minimum in its row or column, and all of the numbers are distinct, we have that

$$r_1 < b_1 < r_2 < b_2 < r_3 < b_3 < \ldots$$

Since the table only has n^2 entries (a finite number), the above infinite sequence must repeat. Thus there is an i with $r_i < r_i$. This contradiction means that the two sets of red and blue numbers must be equal.

$\square$