Mathematics Competition

Problem #3 of five - October 18 to November 1, 2013

A room has the shape of a cube with size $10 \times 10 \times 10$. 2013 butterflies are flying around in this room. Prove that at every instant, three or more butterflies are inside a $1 \times 1 \times 1$ cube in the room. Note: Think of each butterfly as a point.

Direct any questions to Gregory Galperin, OM 3361, or Keith Wolcott, OM 3341

Congratulations to our winners, Jerry Bragg, Marika Rosenberger, and David Stevens.

Solution. There are $10 \times 10 \times 10 = 1,000$ cubic units in the room. Suppose that there were 2 or fewer butterflies in each of these unit cubes at some instant. Then there would be a total of 2,000 or fewer butterflies in the room. This contradiction means that at any instant, there must be at least one of these unit cubes that has three or more butterflies in it.

$\square$

Remark 1: On average, per unit cube, there are $\frac{2013}{1000} = 2.013$ butterflies so the average rounded up to the nearest integer (3 in this case) is the number of butterflies that must be in at least one of the unit cubes.

More generally, if n objects are distributed in m spaces, $\frac{n}{m}$ rounded up to the nearest integer is the number of objects that must be in a least one of the spaces. For example if 2013 butterflies are distributed on the number line between 0 and 100, then $\frac{2013}{100} = 20.13$, so at least one unit interval must contain 21 butterflies.

Remark 2: In the original problem it is tempting to think that since there are 13 extra butterflies, that there are 13 cubic units with 3 butterflies in them, but this may not be true, since all 13 extra butterflies could be in one unit cube.