Mathematics Competition

$25 prize for the best solution for each of 5 problems.
$100 prize for solving the most problems throughout the semester.

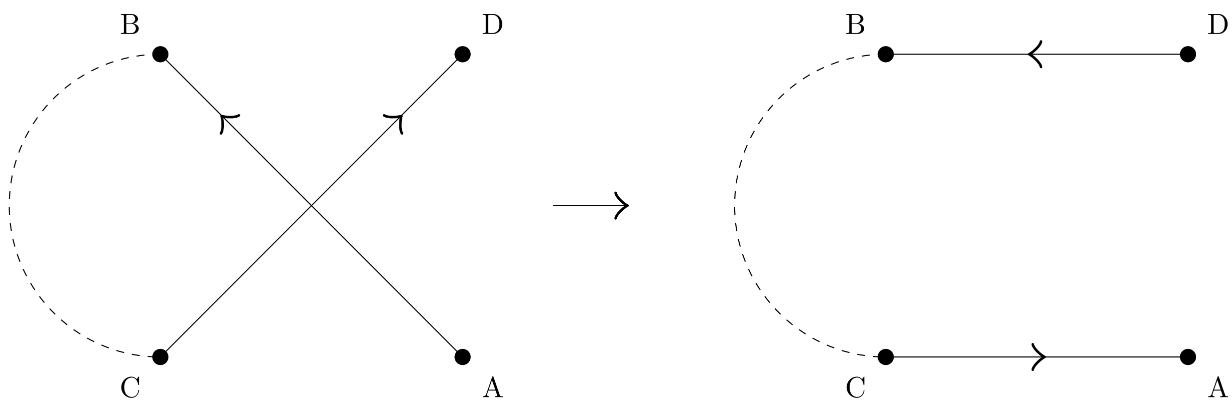
Problem #5 of five - Oct 26 to Nov 16, 2012

Given n points in the plane, no three of which are collinear, prove that a shortest path that contains them all, has no self intersections.

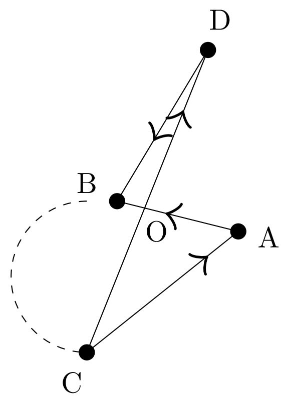
Direct any questions to Kamlesh Parwani, OM 3351, or Keith Wolcott, OM 3341

No complete solutions were submitted to this week's challenge. Joel Blome solved three of the five semester problems and is the $100 winner for the semester.

Solution. Suppose that a shortest path has one or more intersections and consider one of the intersections as in the left figure below. Without loss of generality we put arrows on the segments for the direction of the path. Since it is a a single path, eventually, either B connects to C or D connects to A. Again, without loss of generality we will assume that B connects to C as shown by the dashed curve in the left figure. Now, if we change the path as shown in the right figure below, the new path is still a single path since B connects to C and we will argue that the new path is shorter. This would contradict our original assumption that we had a shortest path with an intersection.



Keep in mind that the four points A, B, C, and D are most likely not arranged in a square as we have depicted them. The figure below shows another possible arrangement where it is not so clear that the new path is shorter.



If O is the intersection point, then by the triangle inequality any one side of a triangle is shorter than the sum of the other two. If we use BD to denote the length of the segment from B to D, then

$$BD < OB + OD$$

and

$$AC < OA + OC.$$

These are strict inequalities since these triangles are not degenerate. This is because we are given that no three points are collinear (for example, if O was on line BD it would mean that C, B, and D would be collinear). Adding these two inequalities gives

$$BD + AC < OB + OD + OA + OC = AB + CD.$$

This shows that the sum of the two new path segments is shorter than the sum of the two original path segments that intersected. This contradicts the fact that we started with a shortest path that had an intersection and therefore all shortest paths have no intersections.

$\square$