Mathematics Competition

\$25 prize for the best solution for each of 5 problems.
\$100 prize for solving the most problems throughout the semester.

Problem #4 of five - Oct 12 to Oct 26, 2012

An old calculus problem asks you to cut a one meter long wire into two pieces and form a regular polygon out of one piece and a circle out of the other piece. The problem is to find the location of the cut that minimizes the total area of the two shapes.

a) If the regular polygon is an equilateral triangle, prove that the minimum occurs when the circle is the size that can be inscribed in the triangle.

b) If the regular polygon is a square, prove that the minimum occurs when the circle is the size that can be inscribed in the square.

c) Is the above true for any regular polygon? Prove or disprove.

Direct any questions to Kamlesh Parwani, OM 3351, or Keith Wolcott, OM 3341

A complete solution was submitted by Joel Blome (\$50 winner).

Solution. Parts a), b), and c) can all be solved by brute force calculation, but usually in mathematics, when a result is very beautiful and seems like a coincidence, it is not a coincidence and understanding why it is true can lead to an easy proof. The brute force solution of this problem involves getting functions for the two areas, say C and P for the areas of the circle and the polygon in terms of perimeter. After adding the two functions, differentiating and setting equal to zero gives the critical point where the minimum total area occurs. Thus if x is the location of the cut on the wire, the perimeter for the polygon is x and the perimeter for the circle is $1-x$. Then the total area function is

$$A(x) = P(x) + C(1-x)$$

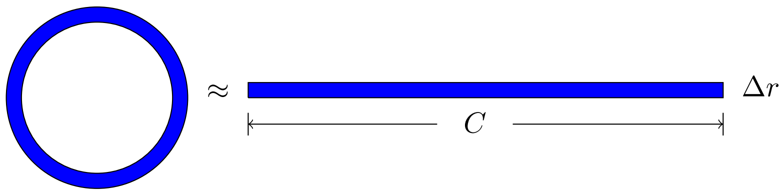
and

$$A'(x) = P'(x) + C'(1-x)(-1) = P'(x) - C'(1-x).$$

Therefore the critical point occurs when the two derivatives are equal if we differentiate each with respect to its perimeter.

Then the question is, what is the rate of change of the area, of a regular polygon or a circle, with respect to perimeter? We will call the radius of a regular polygon, the radius r of the circle that can be inscribed in the regular polygon. Then we will show that the rate of change of the area, of a regular polygon or a circle, in terms of perimeter is equal to the radius r. If this is true, then the solution to parts a), b), and c) is immediate since the critical point is exactly when the two radii are equal, which is when the circle can be inscribed in the polygon. We also need to argue that the minimum occurs at this critical point, but by the physical nature of the problem it is clear that the minimum isn't at the end points of the interval and since this is the only critical point it must be where the minimum occurs.

Now we give an argument that the rate of change of the area, of a circle or regular polygon, in terms of perimeter is equal to the radius. This can be shown by direct calculation, but again, our goal is to give a proof that allows us to understand why it is true. We will argue that the rate of change of area with respect to radius is perimeter and that the rate of change of area with respect to perimeter is radius. As in the figure, if the radius is changed by Δr then the area is changed by approximately $C\Delta r$ where C is the circumference of the circle.



Thus $\Delta A \approx C\Delta r$ or $\frac{\Delta A}{\Delta r} \approx C$ and as Δr approaches zero, $\frac{dA}{dr} = C$.

Now, what we want to compute is $\frac{dA}{dC}$. The important fact to note is that C and r are linearly related. Say, $C = kr$ (here $k = 2\pi$ but the constant will be different for the polygon case). Then $dr = \frac{dC}{k}$ and as above we have $dA = Cdr = (kr)\left(\frac{dC}{k}\right) = rdC$ and thus $\frac{dA}{dC} = r$. Exactly the same method shows that the same is true for a regular polygon (where perimeter is substituted for circumference and r is the radius of an inscribed circle).

Thus we have shown that the minimum occurs exactly when the derivatives of the two functions are equal, where we have differentiated with respect to perimeter. Since these two derivatives both equal their respective radius, they are equal when the two radii are equal, which is when the circle can be inscribed in the polygon.

Note that this proof immediately extends from regular polygons to polygons that have inscribed circles (circles that are tangent to each side of the polygon).

Note also that this proof immediately extends to the more general situation of cutting the wire into $m+1$ pieces and forming a circle out of one piece and regular polygons (with any given number of sides) out of the remaining m pieces. Again the minimum total area is when the rates of change with respect to perimeter are the same for all of the geometric shapes. To see this, set the lengths of the pieces of wire used for the polygons to $x_1, x_2, \ldots, x_m$ with the circle using the remaining wire and let

$$P_1(x_1), \ldots, P_m(x_m)$$

and

$$C(1 - x_1 - x_2 - \ldots - x_m)$$

be the areas of the polygons and the circle. Then if A is the sum of these areas, the partial derivatives set equal to zero are

$$P'(x_k) - C'(1 - x_1 - x_2 - \ldots - x_m) = 0 \qquad \text{for } 1 \leq k \leq m.$$

Thus again, the minimum occurs when these derivatives with respect to perimeter are equal, which is when the radii are all equal, which is when the circle can be inscribed in each of the polygons.