Mathematics Competition

$25 prize for the best solution for each of 5 problems.
$100 prize for solving the most problems throughout the semester.

Problem #3 of five - Sept 28 to Oct 12, 2012

Let n be a positive integer. If 2^n has k digits, how many digits does 5^n have? Express your answer in terms of n and k and prove your answer.

Direct any questions to Kamlesh Parwani, OM 3351, or Keith Wolcott, OM 3341

Partial solutions were submitted to this week's challenge by Amanda Smith and Haylee Beck.

Solution. If 2^n has k digits, then

$$10^{k-1} < 2^n < 10^k. \tag{1}$$

Similarly, if we let d be the number of digits of 5^n, we have

$$10^{d-1} < 5^n < 10^d. \tag{2}$$

The key to a solution is to notice that $2^n \times 5^n = 10^n$, and in order to utilize this observation, multiply inequality 1 and 2. After multiplying these inequalities, we obtain

$$10^{k+d-2} < 10^n < 10^{k+d}.$$

The expression above implies that n, an integer, lies in between the integer $k + d$ and the integer $k + d - 2$. There is only one choice, $n = k + d - 1$. It now immediately follows that $d = n - k + 1$. So 5^n has $n - k + 1$ digits.