Mat 2345 — Discrete Math
Week 13

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Fall 2009

Student Responsibilities — Week 13

- **Reading**: Textbook, Section 8.1 – 8.
- **Assignments**:

  Sec 8.1 Due **Wed** 11/18 : 1a-d, 3a-d, 5ab, 16, 28, 31, 48bd
  Sec 8.3 Due **Thurs** 11/19 : 1ab, 3ab, 5, 14a-c, 18(ab), 23, 26, 36
  Sec 8.5 Due **Mon** 11/30 : 2ad, 5, 15, 22, 35, 43a-c, 61

- **Attendance**: Thankfully Encouraged

Week 13 Overview

- Sec 8.1 Relations and Their Properties
- Sec 8.3 Representing Relations
- Sec 8.5 Equivalence Relations

Section 8.1 — Relations and Their Properties

- **Definition**: A **binary relation R** from a set A to a set B is a subset $R \subseteq A \times B$.
- Note: there are no constraints on relations as there are on functions.
- We have a common graphical representation of relations, a directed graph.

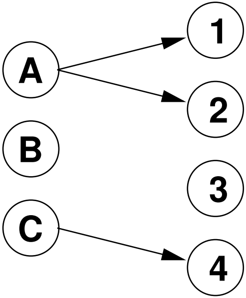
- **Definition**: A **Directed Graph (DiGraph) D** from A to B is:
  1. a collection of **vertices V** $\subseteq A \cup B$, and
  2. a collection of **edges E** $\subseteq A \times B$
- If there is an ordered pair $e =< x, y >$ in R, then there is an **arc** or **edge** from x to y in D. (Note: $E = R$)
- The elements x and y are called the **initial** and **terminal** vertices of the edge e.

Relation Example 1

- Let A = { a, b, c },
- B = { 1, 2, 3, 4 }, and
- R be defined by the ordered pairs or edges:

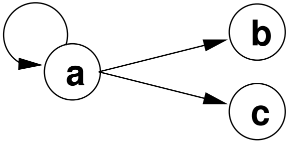
$$\{ < a, 1 >, < a, 2 >, < c, 4 > \}$$

- Then we can represent R by the digraph D:



Relation Example 2

- **Definition**: A binary relation **R on a set A** is a subset of $A \times A$ or a relation from A to A.
- Let A = { a, b, c }
- R = $\{ <a, a>, <a, b>, <a, c> \}$
- Then a digraph representation of R is:



- Note, an arc of the form $<x, x>$ on a digraph is called a **loop**.
- Question: How many binary relations are there on a set A? Another way to look at it: how many subsets are there of A $\times$ A?

Special Properties of Binary Relations

Given

1. A universe U
2. A binary relation R on a subset A of U

- **Definition**: R is **reflexive** IFF

$$\forall\, x \quad [x \in A \rightarrow < x, x > \in R]$$

- Notes:
  - If $A = \emptyset$, then the implication is vacuously true
  - The void relation on an empty set is reflexive
  - If A is not void, then **all** vertices in the reflexive relation must have loops

Symmetric and Antisymmetric Properties

- **Definition**: R is **symmetric** IFF

$$\forall x \forall y [< x, y > \in R \rightarrow < y, x > \in R$$

Note: if there is an arc $< x, y >$, there must be an arc $< y, x >$

- **Definition**: R is **antisymmetric** IFF

$$\forall x \forall y [< x, y > \in R \wedge < y, x > \in R \rightarrow x = y]$$

Note: If there is an arc from x to y, there cannot be one from y to x if x$\neq$ y.

You should be able to show that logically, if $< x, y >$ is in R and x $\neq$ y, then $< y, x >$ is not in R.

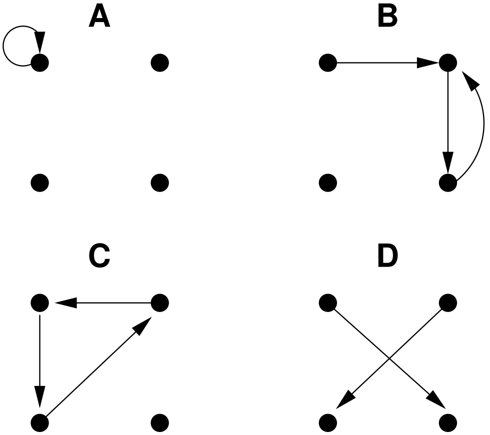
The Transitive Property

- **Definition**: R is **transitive** IFF

$$\forall x \forall y \forall z [< x, y > \in R \wedge < y, z > \in R \rightarrow < x, z > \in R]$$

Note: If there is an arc from x to y and one from y to z, then there must be one from x to z.

This is the most difficult property to check. We will develop algorithms to check this later.



R	reflexive	symmetric	antisymmetric	transitive
A		√	√	√
B				
C			√	
D			√	√

Combining Relations — Set Operations

- A very large set of potential questions! For example, let R1 and R2 be binary relations on a set A. Then we have questions of the form:

  If R1 has Property_1 and
  R2 has Property_2,
  does R1 $\star$ R2 have Property_3?

- For example, If R1 is symmetric and R2 is antisymmetric, does it follow that R1 $\cup$ R2 is transitive?

  If so, prove it. Otherwise, find a counterexample.

Another Example

- Let R1 and R2 be transitive on A. Does it follow that R1 $\cup$ R2 is transitive?
- Consider:
  - A $= \{$ 1, 2 $\}$
  - R1 $= \{ <1,2> \}$
  - R2 $= \{ <2,1> \}$
- Then R1 $\cup$ R2 $= \{ <1,2>, <2,1> \}$. which is **not** transitive. (Why not?)

Composition of Relations

- **Definition**: Suppose
  - R1 is a relation from A to B
  - R2 is a relation from B to C

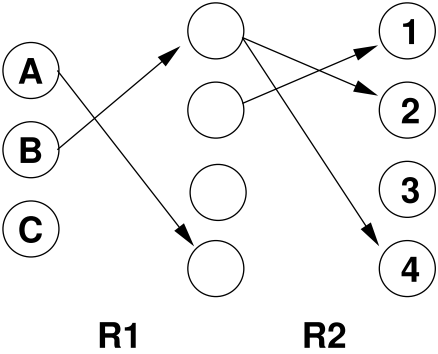
  Then the **composition of R2 with R1**, denoted R2 $\circ$ R1, is the relation from A to C:

  If $< x, y >$ is a member of R1 and
  $< y, z >$ is a member of R2 , then
  $< x, z >$ is a member of R2 $\circ$ R1

- For $< x, y >$ to be in the composite relation R2 $\circ$ R1, there must exist a y in B

- We read compositions right to left as in functions, applying R1 first, then R2

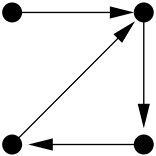
Example of a Composite Relation



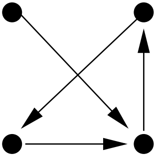
$$R2 \circ R1 = \{ <B, 2>, <B, 4> \}$$

A Relation Composed with Itself

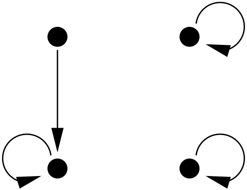
- **Definition**: Let R be a **binary relation** on A. Then the powers $R^n, n = 1, 2, 3, \ldots$ are defined recursively by:
  - **Basis**: $\mathrm{R}^1 = \mathrm{R}$
  - **Induction**: $\mathrm{R}^{n+1} = \mathrm{R}^n \circ \mathrm{R}$
- Note: An ordered pair $< x, y >$ is in R^n IFF there is a **path** of length n from x to y following the arcs (in the direction of the arrows) in R.



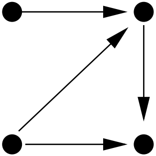
R^1



R^2



R^3



R^4

A Very Important Theorem

R is transitive IFF $R^n \subseteq R$ for $n > 0$.

Proof ($\Rightarrow$): R transitive $\rightarrow R^n \subseteq R$

Use a direct proof with proof by induction

- Assume R is transitive
- Now show $R^n \subseteq R$ by induction
  Basis: Obviously true for $n = 1$
  Induction:
  - IH: Assume $R^k \subseteq R^1$ for some arbitrary $k > 0$
  - IS: Show $R^{k+1} \subseteq R^1$

    $R^{k+1} = R^k \circ R$, so if $< x, y >$ is in R^{k+1}, then there is a z such that $< x, z >$ is in R^k and $< z, y >$ is in R.

    But, since $R^k \subseteq R$, $< x, z >$ is in R

    R is transitive, so $< x, y >$ is in R

    Since $< x, y >$ was an arbitrary edge, the result follows

Proof ($\Leftarrow$): $R^n \subseteq R \rightarrow R$ is transitive

Use the fact that $R^2 \subseteq R$ and the definition of transitivity. Proof left as an exercise. . .

Thus, (given a finished proof of the above) we have shown R is transitive IFF $R^n \subseteq R$ for $n > 0$.

Section 8.3 — Representing Relations

Connection Matrices

- Let R be a relation from A $= \{ a_1, a_2, \ldots, a_m \}$ to B $= \{ b_1, b_2, \ldots, b_n \}$
- **Definition**: An $m \times n$ **connection matrix**, M, for R is defined by:

$$m_{i,j} = \begin{cases} 1 & if \quad < a_i, b_j > \quad \in \quad R \\ 0 & otherwise \end{cases}$$

- **Example**: We assume the rows are labeled with the elements of A and the columns are labeled with the elements of B.

  Let A = { a, b, c }, B = { e, f, g, h }, and

  R $= \{ < a, e >, < c, g > \}$

- Then the connection matrix M for R is: $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$
- Note: The order of the elements of A and B is important!

Theorem. let R be a binary relation on a set A and let M be its connection matrix. Then:

- R is reflexive IFF $M_{i,i} = 1$ for all $1 \leq i \leq |A|$
- R is symmetric IFF M is a symmetric Matrix: $\mathrm{M} = \mathrm{M}^T$
- R is antisymmetric if $M_{ij} = 0$ or $M_{ji} = 0$ for all $\mathrm{i} \neq \mathrm{j}$

Combining Connection Matrices — Join

- **Definition**: The **join** of two matrices, M_1 and M_2, denoted $M_1 \vee M_2$, is the component–wise Boolean "or" of the two matrices.

- **Fact**: If M_1 is the connection matrix for R_1, and M_2 is the connection matrix for R_2, then the join of M_1 and M_2, $M_1 \vee M_2$, is the connection matrix for $R_1 \cup R_2$

Combining Connection Matrices — Meet

- **Definition**:The **meet** of two matrices, M_1 and M_2, denoted $M_1 \wedge M_2$, is the component–wise Boolean "and" of the two matrices.

- **Fact**: If M_1 is the connection matrix for R_1, and M_2 is the connection matrix for R_2, then the meet of M_1 and M_2, $M_1 \wedge M_2$, is the connection matrix for $R_1 \cap R_2$

Given the connection matrix for two relations, how does one find the connection matrix for:

- The complement? $((A \times A) - R)$
- The relative complement? $(R_1 - R_2,\ R_2 - R_1)$
- The symmetric difference? $(R_1 \oplus R_2 = (R_1 - R_2) \cup (R_2 - R_1))$

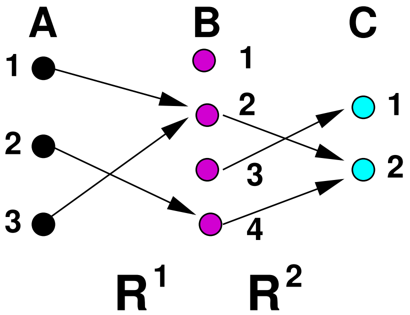
The Composition

- **Definition**: Let M_1 be the connection matrix for R_1, and M_2 be the connection matrix for R_2. Then the **Boolean product** of these two matrices, denoted $M_1 \otimes M_2$, is the connection matrix for the composition of R_2 with R_1, $R_2 \circ R_1$

$$(M_1 \otimes M_2)_{ij} = \vee_{k=1}^{n}[(M_1)_{ik} \wedge (M_2)_{kj})]$$

- Why? In order for there to be an arc $< x, z >$ in the composition, then there must be an arc $< x, y >$ in R_1 and an arc $< y, z >$ in R_2 for some y.
- The Boolean product checks all possible y's. If at least one such path exists, that is sufficient.
- Note: The matrices M_1 and M_2 must be **conformable**: the number of columns of M_1 must equal the number of rows of M_2.
- If M_1 is $m \times n$ and M_2 is $n \times p$, then $M_1 \otimes M_2$ is $m \times p$

Composition Example



$$M_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \quad M_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \quad M_1 \otimes M_2 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$$

$$
\begin{aligned}
(\mathrm{M}_1 \otimes \mathrm{M}_2)_{12} &= [(M_1)_{11} \wedge (M_2)_{12}] \vee \\
&\quad [(M_1)_{12} \wedge (M_2)_{22}] \vee \\
&\quad [(M_1)_{13} \wedge (M_2)_{32}] \vee \\
&\quad [(M_1)_{14} \wedge (M_2)_{42}] \vee \\
&= [0 \wedge 0] \vee [1 \wedge 1] \vee [0 \wedge 0] \vee [0 \wedge 1] \\
&= 1
\end{aligned}
$$

- There is an arc in R_1 from node 1 in A to node 2 in B
- There is an arc in R_2 from node 2 in B to node 2 in C
- Hence, there is an arc in $R_2 \circ R_1$ from node 1 in A to node 2 in C.
- **A useful result**: $M_{R^n} = (M_R)^n$

Digraphs

Given the digraphs for R_1 and R_2, find the digraphs for:

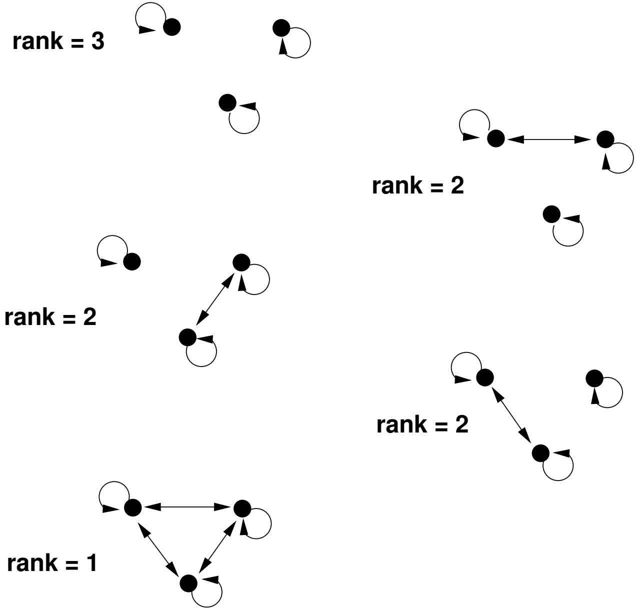
- $R_2 \cup R_1$
- $R_2 \cap R_1$
- $R_2 - R_1$
- $R_2 \otimes R_1$
- complement(R_1)

Section 8.5 — Equivalence Relations

Wherein we define new types of important relations by grouping properties of relations together.

- **Definition**: A relation R on a set A is an **equivalence relation** IFF R is:
  - **reflexive**
  - **symmetric**, and
  - **transitive**
- Equivalence relations are easily recognized in digraphs:
  - The subset of all elements related to a particular element forms a universal relation (i.e., contains all possible arcs) on that subset.
  - The digraph (or subdigraph) representing the subset is called a **complete** digraph (or subdigraph). **All** arcs are present between the included vertices.
  - The number of such subsets is called the **rank** of the equivalence relation.

All Equivalence Relations on a set with 3 Elements



- Each complete subset is called an **equivalence class**.

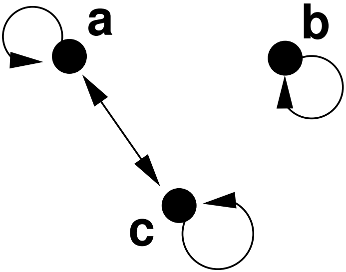
- A bracket around an element means the equivalence class in which the element lies.

$$[x] = \{y| < x, y > \in R\}$$

- The element in the bracket is called a **representative** of the equivalence class — we could have chosen any element in that class.

- There are three ways to say "in the **same equivalence class**":
  1. aRb
  2. $[a] = [b]$
  3. $[a] \cap [b] \neq \emptyset$

Example — Equivalence Classes

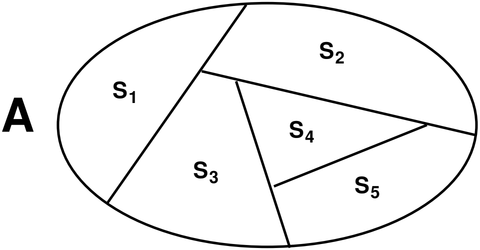


$$[a] = \{a, c\}, \quad [c] = \{a, c\}, \quad [b] = \{b\}$$

$$\text{rank} = 2$$

Partitions

- **Definition**: Let $S_1, S_2, \ldots, S_n$ be a collection of subsets of A. Then the collection forms a **partition** of A if the subsets are **non–empty**, **disjoint**, and **exhaust A**.
  - $S_i \neq \emptyset$
  - $S_i \cap S_j = \emptyset$ if $i \neq j$
  - $\bigcup S_i = A$



Theorem. The equivalence classes of an equivalence relation R **partition** the set A into disjoint nonempty subsets whose union is the entire set.

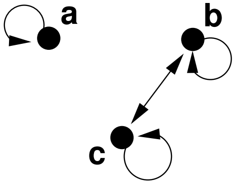
The partition is denoted A/R and is called:

- the **quotient set** or
- the **partition of A induced by R**, or
- **A modulo R**

Examples

1. The set of integers such that aRb IFF a = b or a = −b

2. The natural numbers mod any integer:
   For example, N mod 3 divides the natural numbers into 3 equivalence classes: $[0]_3$, $[1]_3$, $[2]_3$

3. 

$$[a] = \{a\}, \quad [b] = \{b, c\}, \quad [c] = \{b, c\}$$

rank = 2

Theorem. Let R be an equivalence relation on A. Then either

$$[a] = [b]$$

or

$$[a] \cap [b] = \emptyset$$

Why?

Review — $R \subseteq A \times A$

- **reflexive**: $(a, a) \in R \;\; \forall \;\; a \in A$
- **symmetric**: $(b, a) \in R \leftrightarrow (a, b) \in R \;\; for \;\; a, b \in A$
- **antisymmetric**:
  $(b, a) \in R \;\; and \;\; (a, b) \in R, \;\; then \;\; a = b \;\; for \;\; a, b \in A$
- **transitive**:
  $(a, b) \in R \;\; and \;\; (b, c) \in R, \;\; then \; (a, c) \in R \;\; for \;\; a, b, c \in A$

Transitive Closure

- A **path** in a digraph is a sequence of connected edges.
- A path has **length** n, where n is the number of edges in the path.
- A **circuit** or **cycle** is a path that begins and ends at the same vertex.
- The **connectivity relation** $\mathbf{R}^\star$ is the set of all pairs (a, b) such that there is a path between a and b in the relation R.
- This is also called the **transitive closure** of R.

Theorem. If R_1 and R_2 are equivalence relations on A, then $R_1 \cap R_2$ is an equivalence relation on A.

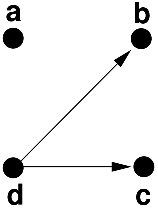
Proof outline. It would suffice to show that the **intersection of**:

- reflexive relations is reflexive
- symmetric relations is symmetric, and
- transitive relations is transitive

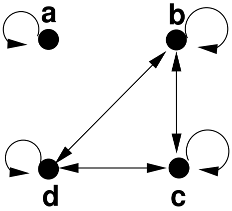
Reflexive, Symmetric, Transitive Closure

Definition. Let R be a relation on A. Then the **reflexive**, **symmetric**, **transitive** closure of R, denoted **tsr(R)**, is an **equivalence relation** induced by R on A.

Example:



R



tsr(R), rank = 2

A = [a] $\cup$ [b] = {a} $\cup$ {b, c, d}

A/R = { {a}, {b, c, d} }

Theorem. tsr(R) is an equivalence relation.

Proof. We must be careful and show that tsr(R) is still symmetric and reflexive.

- Since we only add arcs (rather than delete arcs) when computing closures, it must be that tsr(R) is reflexive since all loops $< x, x >$ on the digraph must be present when constructing r(R).
- If there is an arc $< x, y >$, then the symmetric closure of r(R) ensures there is an arc $< y, x >$.
- We may now argue that if we construct the transitive closure of sr(R) and we add an edge $< x, z >$ because there is a path from x to z, then there must also exist a path from z to x, and hence we also must add an edge $< z, x >$. Hence the transitive closure of sr(R) is symmetric.