

Mat 2345

Chapter Six

Relations

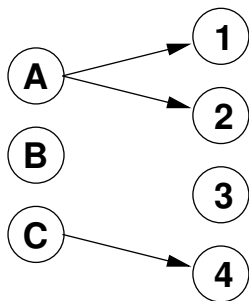
Fall 2007

Section 6.1 — Relations and Their Properties

- ▶ **Definition:** A **binary relation** R from a set A to a set B is a subset $R \subseteq A \times B$.
- ▶ Note: there are no constraints on relations as there are on functions.
- ▶ We have a common graphical representation of relations, a directed graph.
- ▶ **Definition:** A **Directed Graph (DiGraph)** D from A to B is:
 1. a collection of **vertices** $V \subseteq A \cup B$, and
 2. a collection of **edges** $R \subseteq A \times B$
- ▶ If there is an ordered pair $e = \langle x, y \rangle$ in R , then there is an **arc** or **edge** from x to y in D .
- ▶ The elements x and y are called the **initial** and **terminal** vertices of the edge e .

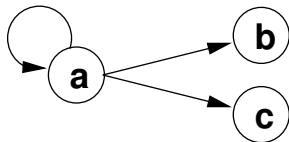
Relation Example 1

- ▶ Let $A = \{ a, b, c \}$,
- ▶ $B = \{ 1, 2, 3, 4 \}$, and
- ▶ R be defined by the ordered pairs or edges:
 $\{ \langle a, 1 \rangle, \langle a, 2 \rangle, \langle c, 4 \rangle \}$
- ▶ Then we can represent R by the digraph D:



Relation Example 2

- ▶ **Definition:** A binary relation **R on a set A** is a subset of $A \times A$ or a relation from A to A.
- ▶ Let $A = \{ a, b, c \}$
- ▶ $R = \{ \langle a, a \rangle, \langle a, b \rangle, \langle a, c \rangle \}$
- ▶ Then a digraph representation of R is:



- ▶ Note, an arc of the form $\langle x, x \rangle$ on a digraph is called a **loop**.
- ▶ Question: How many binary relations are there on a set A?

Special Properties of Binary Relations

Given

1. A universe U
2. A binary relation R on a subset A of U

► **Definition:** R is **reflexive** IFF

$$\forall x \ [x \in U \rightarrow \langle x, x \rangle \in R]$$

► Notes:

- If $U = \emptyset$, then the implication is vacuously true
- The void relation on a void Universe is reflexive
- If U is not void, then **all** vertices in the reflexive relation must have loops

Symmetric and Antisymmetric Properties

- ▶ **Definition:** R is **symmetric** IFF

$$\forall x \forall y [\langle x, y \rangle \in R \rightarrow \langle y, x \rangle \in R]$$

Note: if there is an arc $\langle x, y \rangle$, there must be an arc $\langle y, x \rangle$

- ▶ **Definition:** R is **antisymmetric** IFF

$$\forall x \forall y [\langle x, y \rangle \in R \wedge \langle y, x \rangle \in R \rightarrow x = y]$$

Note: If there is an arc from x to y, there cannot be one from y to x if $x \neq y$.

You should be able to show that logically, if $\langle x, y \rangle$ is in R and $x \neq y$, then $\langle y, x \rangle$ is not in R.

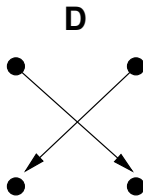
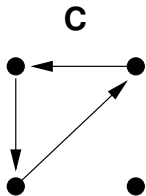
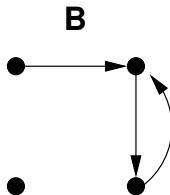
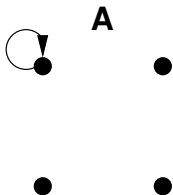
The Transitive Property

- ▶ **Definition:** R is **transitive** IFF

$$\forall x \forall y \forall z [\langle x, y \rangle \in R \wedge \langle y, z \rangle \in R \rightarrow \langle x, z \rangle \in R]$$

Note: If there is an arc from x to y and one from y to z, then there must be one from x to z.

This is the most difficult property to check. We will develop algorithms to check this later.



A not reflexive, symmetric, antisymmetric, transitive

B not reflexive, not symmetric, not antisymmetric, not transitive

C not reflexive, not symmetric, antisymmetric, not transitive

D not reflexive, not symmetric, antisymmetric, transitive

Combining Relations — Set Operations

- ▶ A very large set of potential questions! For example, let R_1 and R_2 be binary relations on a set A . Then we have questions of the form:

If R_1 has Property_1 and
 R_2 has Property_2,
 does $R_1 \star R_2$ have Property_3?

- ▶ For example, If R_1 is symmetric and R_2 is antisymmetric, does it follow that $R_1 \cup R_2$ is transitive?

If so, prove it. Otherwise, find a counterexample.

Another Example

- ▶ Let R_1 and R_2 be transitive on A . Does it follow that $R_1 \cup R_2$ is transitive?
- ▶ Consider:
 - ▶ $A = \{ 1, 2 \}$
 - ▶ $R_1 = \{ \langle 1, 2 \rangle \}$
 - ▶ $R_2 = \{ \langle 2, 1 \rangle \}$
- ▶ Then $R_1 \cup R_2 = \{ \langle 1, 2 \rangle, \langle 2, 1 \rangle \}$. which is **not** transitive. (Why not?)

Composition of Relations

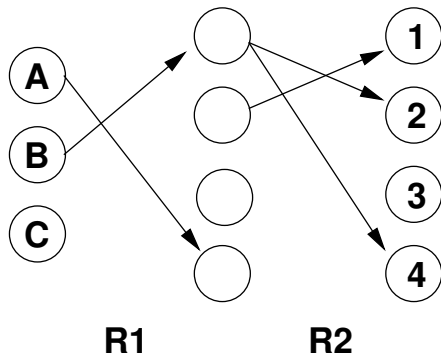
- ▶ **Definition:** Suppose
 - ▶ R_1 is a relation from A to B
 - ▶ R_2 is a relation from B to C

Then the **composition of R_2 with R_1** , denoted $R_2 \circ R_1$, is the relation from A to C :

If $\langle x, y \rangle$ is a member of R_1 and
 $\langle y, z \rangle$ is a member of R_2 , then
 $\langle x, z \rangle$ is a member of $R_2 \circ R_1$

- ▶ For $\langle x, y \rangle$ to be in the composite relation $R_2 \circ R_1$, there must exist a y in B
- ▶ We read compositions right to left as in functions, applying R_1 first, then R_2

Example of a Composite Relation

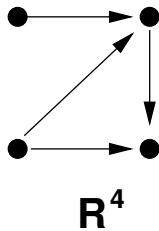
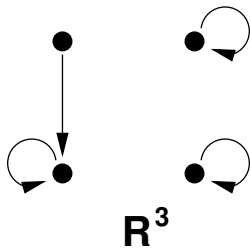
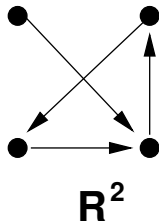
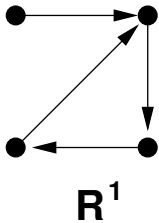


$$R_2 \circ R_1 = \{ \langle B, 2 \rangle, \langle B, 4 \rangle \}$$

A Relation Composed with Itself

- ▶ **Definition:** Let R be a **binary relation** on A . Then
 - ▶ **Basis:** $R^1 = R$
 - ▶ **Induction:** $R^{n+1} = R^n \circ R$

- ▶ **Note:** An ordered pair $\langle x, y \rangle$ is in R^n IFF there is a **path** of length n from x to y following the arcs (in the direction of the arrows) in R .



A Very Important Theorem

R is transitive IFF $R^n \subseteq R$ for $n > 0$.

Proof (\Rightarrow): R transitive $\rightarrow R^n \subseteq R$

Use a direct proof and a proof by induction

- ▶ Assume R is transitive
- ▶ Now show $R^n \subseteq R$ by induction

Basis: Obviously true for $n = 1$

Induction:

- ▶ IH: Assume R^n is transitive for some arbitrary $n > 0$
- ▶ IS: Show R^{n+1} is transitive

$R^{n+1} = R^n \circ R$, so if $\langle x, y \rangle$ is in R^{n+1} , then there is a z such that $\langle x, z \rangle$ is in R^n and $\langle z, y \rangle$ is in R .

But, since $R^n \subseteq R$, $\langle x, z \rangle$ is in R

R is transitive, so $\langle x, y \rangle$ is in R

Since $\langle x, y \rangle$ was an arbitrary edge, the result follows

Proof (\Leftarrow): $R^n \subseteq R \rightarrow R$ is transitive

Use the fact that $R^2 \subseteq R$ and the definition of transitivity. Proof left as an exercise. . .

Thus, (given a finished proof of the above) we have shown R is transitive IFF $R^n \subseteq R$ for $n > 0$.

Section 6.3 — Representing Relations

Connection Matrices

- ▶ Let R be a relation from $A = \{ a_1, a_2, \dots, a_m \}$ to $B = \{ b_1, b_2, \dots, b_n \}$

- ▶ **Definition:** An $m \times n$ **connection matrix**, M , for R is defined by:

$$m_{i,j} = \begin{cases} 1 & \text{if } \langle a_i, b_j \rangle \in R \\ 0 & \text{otherwise} \end{cases}$$

- ▶ **Example:** We assume the rows are labeled with the elements of A and the columns are labeled with the elements of B .

Let $A = \{ a, b, c \}$, $B = \{ e, f, g, h \}$, and

$$R = \{ \langle a, e \rangle, \langle c, g \rangle \}$$

- ▶ Then the connection matrix M for R is:
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

- ▶ Note: The order of the elements of A and B is important!

Theorem. let R be a binary relation on a set A and let M be its connection matrix. Then:

- ▶ R is reflexive IFF $M_{i,i} = 1$ for all i
- ▶ R is symmetric IFF M is a symmetric Matrix: $M = M^T$
- ▶ R is antisymmetric if $M_{ij} = 0$ or $M_{ji} = 0$ for all $i \neq j$

Combining Connection Matrices — Join

- ▶ **Definition:** The **join** of two matrices, M_1 and M_2 , denoted $M_1 \vee M_2$, is the component-wise Boolean "or" of the two matrices.
- ▶ **Fact:** If M_1 is the connection matrix for R_1 , and M_2 is the connection matrix for R_2 , then the join of M_1 and M_2 , $M_1 \vee M_2$, is the connection matrix for $R_1 \cup R_2$

Combining Connection Matrices — Meet

- ▶ **Definition:** The **meet** of two matrices, M_1 and M_2 , denoted $M_1 \wedge M_2$, is the component-wise Boolean "and" of the two matrices.
- ▶ **Fact:** If M_1 is the connection matrix for R_1 , and M_2 is the connection matrix for R_2 , then the join of M_1 and M_2 , $M_1 \wedge M_2$, is the connection matrix for $R_1 \cap R_2$

Given the connection matrix for two relations, how does one find the connection matrix for:

- ▶ The complement?
- ▶ The relative complement?
- ▶ The symmetric difference?

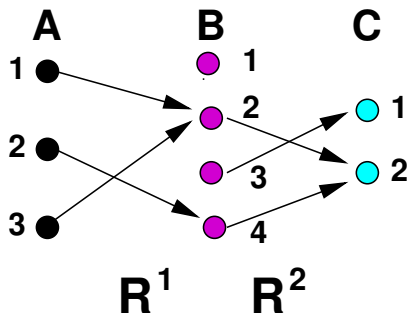
The Composition

- ▶ **Definition:** Let M_1 be the connection matrix for R_1 , and M_2 be the connection matrix for R_2 . Then the **Boolean product** of these two matrices, denoted $M_1 \otimes M_2$, is the connection matrix for the composition of R_2 with R_1 , $R_2 \circ R_1$

$$(M_1 \otimes M_2)_{ij} = \bigvee_{k=1}^n [(M_1)_{ik} \wedge (M_2)_{kj}]$$

- ▶ Why? In order for there to be an arc $\langle x, z \rangle$ in the composition, then there must be an arc $\langle x, y \rangle$ in R_1 and an arc $\langle y, z \rangle$ in R_2 for some y .
- ▶ The Boolean product checks all possible y 's. If at least one such path exists, that is sufficient.
- ▶ Note: The matrices M_1 and M_2 must be **conformable**: the number of columns of M_1 must equal the number of rows of M_2 .
- ▶ If M_1 is $m \times n$ and M_2 is $N \times p$, then $M_1 \otimes M_2$ is $m \times p$

Composition Example



$$M_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$M_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$M_1 \otimes M_2 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$$

$$\begin{aligned}
(M_1 \otimes M_2)_{12} &= [(M_1)_{11} \wedge (M_2)_{12}] \vee \\
&\quad [(M_1)_{12} \wedge (M_2)_{22}] \vee \\
&\quad [(M_1)_{13} \wedge (M_2)_{32}] \vee \\
&\quad [(M_1)_{14} \wedge (M_2)_{42}] \vee \\
&= [0 \wedge 0] \vee [1 \wedge 1] \vee [0 \wedge 0] \vee [0 \wedge 1] \\
&= 1
\end{aligned}$$

- ▶ There is an arc in R_1 from node 1 in A to node 2 in B
- ▶ There is an arc in R_2 from node 2 in B to node 2 in C
- ▶ Hence, there is an arc in $R_2 \circ R_1$ from node 1 in A to node 2 in C.
- ▶ **A useful result:** $M_{R^n} = M_R^n$

Digraphs

Given the digraphs for R_1 and R_2 , find the digraphs for:

▶ $R_2 \cup R_1$

▶ $R_2 \cap R_1$

▶ $R_2 - R_1$

▶ $R_2 \otimes R_1$

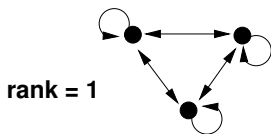
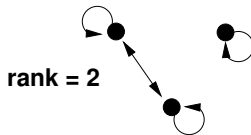
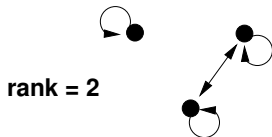
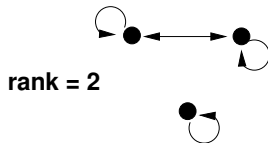
▶ $\text{complement}(R_1)$

Section 6.5 — Equivalence Relations

Wherein we define new types of important relations by grouping properties of relations together.

- ▶ **Definition:** A relation R on a set A is an **equivalence relation** IFF R is:
 - ▶ reflexive
 - ▶ symmetric, and
 - ▶ transitive
- ▶ Equivalence relations are easily recognized in digraphs:
 - ▶ The subset of all elements related to a particular element forms a universal relation (i.e., contains all possible arcs) on that subset.
 - ▶ The digraph (or subdigraph) representing the subset is called a **complete** digraph (or subdigraph). **All** arcs are present.
 - ▶ The number of such subsets is called the **rank** of the equivalence relation.

Examples — A has 3 Elements

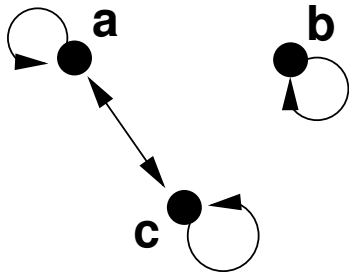


- ▶ Each of the subsets is called an **equivalence class**.
- ▶ A bracket around an element means the equivalence class in which the element lies.

$$[x] = \{y \mid x, y \in R\}$$

- ▶ The element in the bracket is called a **representative** of the equivalence class — we could have chosen any element in that class.
- ▶ There are three ways to say "in the same equivalence class":
 1. aRb
 2. $[a] = [b]$
 3. $[a] \cap [b] \neq \emptyset$

Example — Equivalence Classes

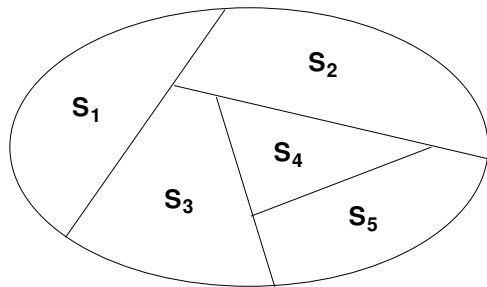


$$[a] = \{a, c\}, \quad [c] = \{a, c\}, \quad [b] = \{b\}$$

$$\text{rank} = 2$$

Partitions

- **Definition:** Let S_1, S_2, \dots, S_n be a collection of subsets of A . Then the collection forms a **partition** of A if the subsets are **non-empty**, **disjoint**, and **exhaust A** .
- $S_i \neq \emptyset$
 - $S_i \cap S_j = \emptyset$ if $i \neq j$
 - $\cup S_i = A$



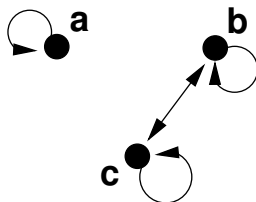
Theorem. The equivalence classes of an equivalence relation R **partition** the set A into disjoint nonempty subsets whose union is the entire set.

The partition is denoted A/R and is called:

- ▶ the **quotient set** or
- ▶ the **partition of A induced by R** , or
- ▶ **A modulo R**

Examples

- ▶ The set of integers such that aRb IFF $a = b$ or $a = -b$
- ▶ The natural numbers mod any integer:
For example, $\mathbb{N} \bmod 3$ divides the natural numbers into 3 equivalence classes: $[0]_3, [1]_3, [2]_3$



$$[a] = \{a, c\}, \quad [c] = \{a, c\}, \quad [b] = \{b\}$$

$$\text{rank} = 2$$

Theorem. Let R be an equivalence relation on A . Then either

$$[a] = [b]$$

or

$$[a] \cap [b] = \emptyset$$

Why?

▶ **reflexive:** $(a, a) \in R \quad \forall a \in A$

▶ **symmetric:** $(b, a) \in R \leftrightarrow (a, b) \in R$ for $a, b \in A$

▶ **antisymmetric:**

$(b, a) \in R$ and $(a, b) \in R$, then $a = b$ for $a, b \in A$

▶ **transitive:**

$(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$ for $a, b, c \in A$

Transitive Closure

- ▶ A **path** in a digraph is a sequence of connected edges.
- ▶ A path has **length** n , where n is the number of edges in the path.
- ▶ A **circuit** or **cycle** is a path that begins and ends at the same vertex.
- ▶ The **connectivity relation** R^* is the set of all pairs (a, b) such that there is a path between a and b in the relation R .
- ▶ This is also called the **transitive closure** of R .

Theorem. If R_1 and R_2 are equivalence relations on A , then $R_1 \cap R_2$ is an equivalence relation on A .

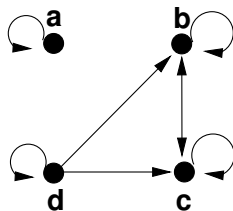
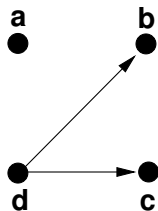
Proof outline. It would suffice to show that the **intersection of:**

- ▶ reflexive relations is reflexive
- ▶ symmetric relations is symmetric, and
- ▶ transitive relations is transitive

Reflexive, Symmetric, Transitive Closure

Definition. Let R be a relation on A . Then the **reflexive, symmetric, transitive** closure of R , denoted $\text{tsr}(R)$, is an **equivalence relation** induced by R on A .

Example:



$\text{tsr}(R)$, rank = 2

$$A = [a] \cup [b] = \{a\} \cup \{b, c, d\}$$

$$A/R = \{ \{a\}, \{b, c, d\} \}$$

Theorem. $\text{tsr}(R)$ is an equivalence relation.

Proof. We must be careful and show that $\text{tsr}(R)$ is still symmetric and reflexive.

- ▶ Since we only add arcs (rather than delete arcs) when computing closures, it must be that $\text{tsr}(R)$ is reflexive since all loops $\langle x, x \rangle$ on the digraph must be present when constructing $r(R)$.
- ▶ If there is an arc $\langle x, y \rangle$, then the symmetric closure of $r(R)$ ensures there is an arc $\langle y, x \rangle$.
- ▶ We may now argue that if we construct the transitive closure of $sr(R)$ and we add an edge $\langle x, z \rangle$ because there is a path from x to z , then there must also exist a path from z to x , and hence we also must add an edge $\langle z, x \rangle$. Hence the transitive closure of $sr(R)$ is symmetric.