Mat 2345
Chapter Five

Recurrence Relations

Fall 2007

Section 5.1 — Algorithmic Complexity and Recurrence Relations

- A **Recurrence Relation** is a way to define a function by an expression involving the same function.

- Ex: Fibonacci Numbers.
  `F(0) = 1,   F(1) = 1,`
  `F(n) = F(n-1) + F(n-2)`

- If we wish to compute the 120^{th} Fibonacci Number, `F(120)`, we could compute `F(0), F(1), F(2), ...F(118)`, and `F(119)` to arrive at `F(120)`.

- Thus, to compute `F(k)` in this manner would take `k` steps.

Fibonacci Closed Form Expression

- It would be more convenient, not to mention more efficient, to have an `explicit` or `closed form` expression to compute `F(n)`.
- Actually, for Fibonacci numbers, it's:

$$F(n) \;=\; \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n, \forall \textit{ natural numbers } n \geq 1$$

Solving Recurrence Relations

There are many methods for solving recurrence relations for their explicit or closed form solution. Here are three:

1. **Substitution Method** — guess a bound and prove it correct by mathematical induction

2. **Iteration Method** (used on Recurrence Relations with Full History, i.e., the n^{th} term depends on *all* the previous values of a function.)

   Idea: convert the recurrence to a summation and rely on summation bounding techniques to solve the recurrence

3. **Master Method** (aka, Divide and Conquer).
   - provides a "cookbook" or formula
   - Recurrence **must** have the form:
     $$T(n) = aT(\tfrac{n}{b}) \; + \; f(n)$$
     where $a \geq 1$, $b > 1$, and $f(n)$ will have some restrictions

The Master Method

Recurrences with the form $T(n) = aT(\frac{n}{b}) \ + \ f(n)$ occur when we have an algorithm that

- **DIVIDES** a problem of size **n** into **a** subproblems,
- each of size $\frac{n}{b}$,
- with a cost of dividing and recombining of $f(n)$.

Note: $\frac{n}{b}$ can mean either $\lfloor \frac{n}{b} \rfloor$ or $\lceil \frac{n}{b} \rceil$

Theorem. The solution of the recurrence relation $T(n) = aT(\frac{n}{b}) + cn^k$, where a and b are integer constants, $a \geq 1$ and $b \geq 2$, and c and k are positive constants, is:

$$T(n) = \begin{cases} O(n^{\log_b(a)}) & \text{if} \quad a > b^k \\ O(n^k \log_b(n)) & \text{if} \quad a = b^k \\ O(n^k) & \text{if} \quad a < b^k \end{cases}$$

Note: We are using a weak form of the Master Method in this course. Another form exists which doesn't restrict the form of $f(n)$ to cn^k.

Applying the Master Method

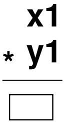
- $T(n) = 9T(\frac{n}{3}) + 4n^6$, assuming $n = 3^m, \; m \geq 1$
  Thus, the algorithm divides the problem into 9 subproblems, each of size $\frac{1}{3}n$, with a cost of dividing and recombining of $4n^6$.
  `a =` `b =` `c =` `k =`

- $T(n) = 7T(\frac{n}{2}) + 18n^2$, assuming $n = 2^m, \; m \geq 1$

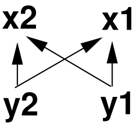
- $T(n) = 2T(\frac{n}{2}) + n + 1$, assuming $n = 2^m, \ m \geq 1$

- $T(n) = 2T(\frac{n}{2}) + n \log n$, assuming $n = 2^m, \ m \geq 1$

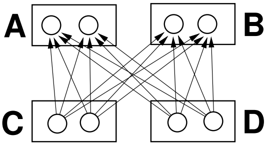
Multiplying n–digit Numbers



Single Multiply
$1 \times 1 = 1$



Two–Digit Multiply
$2 \times 2 = 4$



Four–Digit Multiply
$4 \times 4 = 16$

1. The time it takes to multiply n digits is $4 \times$ the time it takes to multiply $\frac{n}{2}$ digits

2. Then we must sum the products — let's say we can do this in $2n$ time.

Thus, in this divide and conquer strategy, we:

1. create 4 subproblems,
2. each half the size of the original ($\frac{1}{2}n$),
3. with cost $2n$ of dividing and recombining.

Hence, $T(n) = 4T(\frac{n}{2}) + 2n$

`a =` `b =` `c =` `k =`

Section 5.2 — Solving Recurrence Relations

- Recurrence relations which express the terms of a sequence as a **linear combination of previous terms** can be explicitly solved in a systematic way.

- **Definition** A **linear homogeneous recurrence relation of degree k with constant coefficients** is a recurrence relation of the form:

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

where $c_1, c_2, \ldots, c_k$ are real numbers, and $c_k \neq 0$

- **Linear**: the right–hand side is a sum of multiples of the previous terms of the sequence.
- **Homogeneous**: no terms occur that are **not** multiples of the a_j's
- **Coefficients**: all of the terms of the sequence are constants (rather than functions dependent on n)
- **Degree**: is k because a_n is expressed in terms of the previous k terms of the sequence.

A sequence satisfying the recurrence relation in the definition is uniquely determined by this recurrence relation and the k initial conditions:

$$a_0 = C_0, \qquad a_1 = C_1, \qquad \ldots, \qquad a_{k-1} = C_{k-1},$$

Examples of linear homogeneous recurrence relations:

$P_n = 3P_{n-1}$	degree one
$f_n = f_{n-1} + f_{n-2}$	degree two
$a_n = a_{n-5}$	degree five

Examples which are **not** linear homogeneous recurrence relations:

$a_n = a_{n-1} + a_{n-2}^2$	not linear
$H_n = 2H_{n-1} + 2$	not homogeneous
$B_n = nB_{n-5}$	doesn't have constant coefficient

Solving Linear Homogeneous Recurrence — Relations with Constant Coefficients

Idea: look for solutions of the form $a_n = r^n$, where r is a constant.
Note: $a_n = r^n$ is a solution of the recurrence relation:

$$a_n \; = \; c_1 a_{n-1} \; + \; c_2 a_{n-2} \; + \; \ldots \; + \; c_k a_{n-k}$$

if and only if

$$r^n \; = \; c_1 r^{n-1} \; + \; c_2 r^{n-2} \; + \; \ldots \; + \; c_k r^{n-k}$$

Now, divide both sides of the equation by r^{n-k}, and subtract the right–hand side from the left:

$$r^k \; - \; c_1 r^{k-1} \; - \; c_2 r^{k-2} \; - \; \ldots \; - \; c_{k-1} r \; - \; c_k \; = \; 0$$

This is the **characteristic equation** of the recurrence relation.

Note: The sequence $\{a_n\}$ with $a_n = r^n$ is a solution IFF r is a solution to the characteristic equation.

Characteristic Roots

The solutions of the characteristic equation are called the **characteristic roots** of the recurrence relation.

They can be used to create an explicit formula for all the solutions of the recurrence relation.

Theorem 1. Let c_1 and c_2 be real numbers. Suppose that

$$r^2 - c_1 r - c_2 = 0$$

has two distinct roots, r_1 and r_2. Then the sequence $\{a_n\}$ is a solution of the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2}$$

if and only if

$$a_n = \propto_1 r_1^n + \propto_2 r_2^n$$

for $n = 0, 1, 2, \ldots,$ where $\propto_1$ and $\propto_2$ are constants

Solving Recurrence Relations, Example I

$$a_0 = 2, \quad a_1 = 7, \textbf{ and } \quad a_n = a_{n-1} + 2a_{n-2}$$

We see that $c_1 = 1$ and $c_2 = 2$

Characteristic Equation: $r^2 - r - 2 = 0$

Roots: $r = 2$ and $r = -1$

Thus, the sequence $\{a_n\}$ is a solution to the recurrence relation IFF

$$a_n = \propto_1 2^n + \propto_2 (-1)^n$$

for some constants $\propto_1$ and $\propto_2$

Solving Recurrence Relations, Example I — Cont.

From the initial conditions, it follows that:

$$\begin{aligned} a_0 &= 2 = \propto_1 (2^0) + \propto_2 (-1)^0 \\ a_1 &= 7 = \propto_1 (2^1) + \propto_2 (-1)^1 \end{aligned}$$

Solving these two equations yields:

$$\propto_1 = 3 \quad \text{and} \quad \propto_2 = -1$$

Therefore, the **solution** to the recurrence relation and initial conditions is the sequence $\{a_n\}$ with:

$$a_n = 3(2)^n - (-1)^n$$

Solving Recurrence Relations, Example II

$$F_0 = 0, \quad F_1 = 1, \textbf{ and } \quad F_n = F_{n-1} + F_{n-2}$$

We see that $c_1 = 1$ and $c_2 = 1$

Characteristic Equation: $r^2 - r - 1 = 0$

Roots: $r = \frac{1+\sqrt{5}}{2}$ and $r = \frac{1-\sqrt{5}}{2}$

Thus, it follows that the Fibonacci numbers are given by

$$F_n = \propto_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + \propto_2 \left(\frac{1-\sqrt{5}}{2}\right)^n$$

for some constants $\propto_1$ and $\propto_2$

Solving Recurrence Relations, Example II — Cont.

From the initial conditions, it follows that:

$$\begin{array}{lllll} F_0 & = & 0 & = & \alpha_1 + \alpha_2 \\ F_1 & = & 1 & = & \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right) + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right) \end{array}$$

Solving these two equations yields:

$$\alpha_1 = \frac{1}{\sqrt{5}} \text{ and } \alpha_2 = -\frac{1}{\sqrt{5}}$$

Therefore, the **solution** to the recurrence relation and initial conditions is the sequence $\{F_n\}$ with:

$$F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

Solving Recurrence Relations, Example III

$$a_0 = 1, \quad a_1 = 1, \textbf{ and } \quad a_n = 2a_{n-1} + 3a_{n-2}$$

We see that $c_1 = 2$ and $c_2 = 3$

Characteristic Equation: $r^2 - 2r - 3 = 0$

Roots: $r = 3$ and $r = -1$

Thus, the sequence $\{a_n\}$ is a solution to the recurrence relation IFF

$$a_n = \propto_1 3^n + \propto_2 (-1)^n$$

for some constants $\propto_1$ and $\propto_2$

Solving Recurrence Relations, Example III — Cont.

From the initial conditions, it follows that:

$$\begin{aligned} a_0 &= 1 = \propto_1 + \propto_2 \\ a_1 &= 1 = \propto_1 (3) + \propto_2 (-1) \end{aligned}$$

Solving these two equations yields: $\propto_1 = \frac{1}{2}$ and $\propto_2 = \frac{1}{2}$

Therefore, the **solution** to the recurrence relation and initial conditions is the sequence $\{a_n\}$ with:

$$a_n = \frac{1}{2}(3)^n + \frac{1}{2}(-1)^n$$

Solving Recurrence Relations, Example IV

$$a_0 = 1, \quad a_1 = -2, \textbf{ and } \quad a_n = 5a_{n-1} - 6a_{n-2}$$

We see that $c_1 = 5$ and $c_2 = -6$

Characteristic Equation: $r^2 - 5r + 6 = 0$

Roots: $r = 2$ and $r = 3$

Thus, the sequence $\{a_n\}$ is a solution to the recurrence relation IFF

$$a_n = \propto_1 2^n + \propto_2 3^n$$

for some constants $\propto_1$ and $\propto_2$

Solving Recurrence Relations, Example IV — Cont.

From the initial conditions, it follows that:

$$\begin{aligned} a_0 &= 1 &= \propto_1 + \propto_2 \\ a_1 &= \text{-2} &= \propto_1 (2) + \propto_2 (3) \end{aligned}$$

Solving these two equations yields:

$\propto_1 = 5$ and $\propto_2 = -4$

Therefore, the **solution** to the recurrence relation and initial conditions is the sequence $\{a_n\}$ with:

$$a_n = 5(2)^n - 4(3)^n$$

Solving Recurrence Relations, Example V

$$a_0 = 0, \quad a_1 = 1, \textbf{ and } \quad a_n = a_{n-1} + 6a_{n-2}$$

We see that $c_1 = 1$ and $c_2 = 6$

Characteristic Equation: $r^2 - r - 6 = 0$

Roots: $r = 3$ and $r = -2$

Thus, the sequence $\{a_n\}$ is a solution to the recurrence relation IFF

$$a_n = \propto_1 3^n + \propto_2 (-2)^n$$

for some constants $\propto_1$ and $\propto_2$

Solving Recurrence Relations, Example V

From the initial conditions, it follows that:

$$\begin{aligned} a_0 &= 0 = \alpha_1 + \alpha_2 \\ a_1 &= 1 = \alpha_1 (3) + \alpha_2 (-2) \end{aligned}$$

Solving these two equations yields:

$$\alpha_1 = \frac{1}{5} \quad \text{and} \quad \alpha_2 = -\frac{1}{5}$$

Therefore, the **solution** to the recurrence relation and initial conditions is the sequence $\{a_n\}$ with:

$$a_n = \frac{1}{5}(3)^n - \frac{1}{5}(-2)^n$$

What To Do When There's Only One Root?

Theorem 1 **does not apply** when there is a **single** characteristic root of multiplicity two.

Theorem 2. Let c_1 and c_2 be real numbers. Suppose that

$$r^2 - c_1 r - c_2 = 0$$

has only one root, r_0. Then the sequence $\{a_n\}$ is a solution of the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2}$$

if and only if

$$a_n = \propto_1 r_0^n + \propto_2 n r_0^n$$

for $n = 0, 1, 2, \ldots$, where $\propto_1$ and $\propto_2$ are constants

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Notice the **extra factor** of n in the second term!

Single Root, Example I

$$a_0 = 1, \quad a_1 = 6, \textbf{ and } \quad a_n = 6a_{n-1} - 9a_{n-2}$$

We see that $c_1 = 6$ and $c_2 = -9$

Characteristic Equation: $r^2 - 6r + 9 = 0$

Root: $r = 3$ with multiplicity 2

Thus, the sequence $\{a_n\}$ is a solution to the recurrence relation IFF

$$a_n = \propto_1 3^n + \propto_2 n(3)^n$$

for some constants $\propto_1$ and $\propto_2$

Single Root, Example I — Cont.

From the initial conditions, it follows that:

$$\begin{aligned} a_0 &= 1 = \propto_1 \\ a_1 &= 6 = \propto_1 (3) + \propto_2 (3) \end{aligned}$$

Solving these two equations yields: $\propto_1 = 1$ and $\propto_2 = 1$

Therefore, the **solution** to the recurrence relation and initial conditions is the sequence $\{a_n\}$ with:

$$a_n = (3)^n + n(3)^n$$

Single Root, Example II

$$a_0 = 1, \quad a_1 = 3, \textbf{ and } \quad a_n = 4a_{n-1} - 4a_{n-2}$$

We see that $c_1 = 4$ and $c_2 = -4$

Characteristic Equation: $r^2 - 4r + 4 = 0$

Root: $r = 2$ with multiplicity 2

Thus, the sequence $\{a_n\}$ is a solution to the recurrence relation IFF

$$a_n = \propto_1 2^n + \propto_2 n2^n$$

for some constants $\propto_1$ and $\propto_2$

Single Root, Example II — Cont.

From the initial conditions, it follows that:

$$\begin{aligned} a_0 &= 1 = \propto_1 \\ a_1 &= 3 = \propto_1 (2) + \propto_2 (2) \end{aligned}$$

Solving these two equations yields: $\propto_1 = 1$ and $\propto_2 = \frac{1}{2}$

Therefore, the **solution** to the recurrence relation and initial conditions is the sequence $\{a_n\}$ with:

$$a_n = 2^n + \frac{1}{2} n 2^n = 2^n + n 2^{n-1}$$

Single Root, Example III

$$a_0 = 1, \quad a_1 = 12, \textbf{ and } \quad a_n = 8a_{n-1} - 16a_{n-2}$$

We see that $c_1 = 8$ and $c_2 = -16$

Characteristic Equation: $r^2 - 8r + 16 = 0$

Root: $r = 4$ with multiplicity 2

Thus, the sequence $\{a_n\}$ is a solution to the recurrence relation IFF

$$a_n = \propto_1 4^n + \propto_2 n4^n$$

for some constants $\propto_1$ and $\propto_2$

Single Root, Example III — Cont.

From the initial conditions, it follows that:

$$\begin{aligned} a_0 &= 1 &&= \propto_1 \\ a_1 &= 12 &&= \propto_1 (4) + \propto_2 (4) \end{aligned}$$

Solving these two equations yields: $\propto_1 = 1$ and $\propto_2 = 2$

Therefore, the **solution** to the recurrence relation and initial conditions is the sequence $\{a_n\}$ with:

$$a_n = (4)^n - 2n(4)^n$$

Single Root, Example IV

$$a_0 = 2, \quad a_1 = 5, \textbf{ and } \quad a_n = 2a_{n-1} - a_{n-2}$$

We see that $c_1 = 2$ and $c_2 = -1$

Characteristic Equation: $r^2 - 2r + 1 = 0$

Root: $r = 1$ with multiplicity 2

Thus, the sequence $\{a_n\}$ is a solution to the recurrence relation IFF

$$a_n = \propto_1 1^n + \propto_2 n(1)^n$$

for some constants $\propto_1$ and $\propto_2$

Single Root, Example IV — Cont.

From the initial conditions, it follows that:

$$
\begin{aligned}
a_0 &= 2 = \alpha_1 \\
a_1 &= 5 = \alpha_1\,(1) + \alpha_2\,(1)
\end{aligned}
$$

Solving these two equations yields: $\alpha_1 = 2$ and $\alpha_2 = 3$

Therefore, the **solution** to the recurrence relation and initial conditions is the sequence $\{a_n\}$ with:

$$a_n = 2(1)^n + 3n(1)^n = 2 + 3n$$

Solving Recurrence Relations

Definition. A **linear homogeneous recurrence relation of degree k with constant coefficients** is a recurrence relation of the form:

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$$

where $c_1, c_2, \ldots, c_k$ are real numbers, and $c_k \neq 0$.

Theorem 3. Let $c_1, c_2, \ldots, c_k$ be real numbers. Suppose that the characteristic equation

$$r^k - c_1 r^{k-1} - \ldots - c_k = 0$$

has k distinct roots, $r_1, r_2, \ldots, r_k$. Then the sequence $\{a_n\}$ is a solution of the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$$

if and only if

$$a_n = \propto_1 r_1^n + \propto_2 r_2^n + \ldots + \propto_k r_k^n$$

for $n = 0, 1, 2, \ldots,$ where $\propto_1, \propto_2, \ldots, \propto_k$ are constants

Multiple Distinct Roots, Example I

$a_0 = 2$, $a_1 = 5$, $a_2 = 15$, **and** $a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$

We see that $c_1 = 6$, $c_2 = -11$, and $c_3 = 6$

Characteristic Equation:
$r^3 - 6r^2 + 11r - 6 = (r-1)(r-2)(r-3) = 0$

Roots: $r = 1$, $r = 2$, and $r = 3$

Thus, the sequence $\{a_n\}$ is a solution to the recurrence relation IFF

$$a_n = \propto_1 1^n + \propto_2 2^n + \propto_3 3^n$$

for some constants $\propto_1$, $\propto_2$, and $\propto_3$

Multiple Distinct Roots, Example I — Cont.

From the initial conditions, it follows that:

$$\begin{array}{lllllllll} a_0 & = & 2 & = & \propto_1 & + & \propto_2 & + & \propto_3 \\ a_1 & = & 5 & = & \propto_1 & + & \propto_2(2) & + & \propto_3(3) \\ a_2 & = & 15 & = & \propto_1 & + & \propto_2(4) & + & \propto_3(9) \end{array}$$

Solving: $\propto_1 = 1$, $\propto_2 = -1$, and $\propto_3 = 2$

Therefore, the **solution** to the recurrence relation and initial conditions is the sequence $\{a_n\}$ with:

$$a_n = 1 - 2^n + 2(3)^n.$$

Multiple Distinct Roots, Example II

$a_0 = 4$, $a_1 = -9$, $a_2 = -9$, **and** $a_n = 4a_{n-1} - a_{n-2} - 6a_{n-3}$

We see that $c_1 = 4$, $c_2 = -1$, and $c_3 = -6$

Characteristic Equation:
$r^3 - 4r^2 + r + 6 = (r+1)(r-2)(r-3) = 0$

Roots: $r = -1$, $r = 2$, and $r = 3$

Thus, the sequence $\{a_n\}$ is a solution to the recurrence relation IFF

$$a_n = \propto_1 (-1)^n + \propto_2 2^n + \propto_3 3^n$$

for some constants $\propto_1$, $\propto_2$, and $\propto_3$

Multiple Distinct Roots, Example II — Cont.

From the initial conditions, it follows that:

$$
\begin{aligned}
a_0 &= 4 &&= \propto_1 (-1)^0 + \propto_2 2^0 + \propto_3 3^0 \\
& &&= \propto_1 + \propto_2 + \propto_3 \\
a_1 &= \text{-9} &&= \propto_1 (-1)^1 + \propto_2 2^1 + \propto_3 3^1 \\
& &&= -\propto_1 + 2\propto_2 + 3\propto_3 \\
a_2 &= \text{-9} &&= \propto_1 (-1)^2 + \propto_2 2^2 + \propto_3 3^2 \\
& &&= \propto_1 + 4\propto_2 + 9\propto_3
\end{aligned}
$$

Multiple Distinct Roots, Example II — Cont.

Solving: $\propto_1 = 5$, $\propto_2 = 1$, and $\propto_3 = -2$

Therefore, the **solution** to the recurrence relation and initial conditions is the sequence $\{a_n\}$ with:

$$a_n = 5(-1)^n + 2^n - 2(3)^n.$$

Solutions to General Recurrence Relations

The next theorem states the most general result about linear homogeneous recurrence relations with constant coefficients, allowing the characteristic equation to have **multiple** roots.

Key Point: for each root r of the characteristic equation, the general solution has a summand of the form $P(n)r^n$**,** where $P(n)$ is a polynomial of degree $m - 1$, with m **the multiplicity of this root**.

Theorem 4. Let $c_1,\ c_2,\ \ldots,\ c_k$ be real numbers. Suppose that the characteristic equation

$$r^k - c_1 r^{k-1} - \ldots - c_k = 0$$

- has t distinct roots, $r_1,\ r_2,\ \ldots,\ r_t$, with
- multiplicities $m_1, m_2, \ldots, m_t$, respectively, so
- $m_i \geq 1$ for $i = 1, 2, \ldots, t$, and
- $m_1 + m_2 + \ldots + m_t = k$.

Then a sequence $\{a_n\}$ is a solution of the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$$

if and only if

$$\begin{array}{lll} a_n & = & (\propto_{1,0} + \propto_{1,1} n + \ldots + \propto_{1,m_1-1} n^{m_1-1}) r_1^n \\ & + & (\propto_{2,0} + \propto_{2,1} n + \ldots + \propto_{2,m_2-1} n^{m_2-1}) r_2^n \\ & + & \ldots \\ & + & (\propto_{t,0} + \propto_{t,1} n + \ldots + \propto_{t,m_t-1} n^{m_t-1}) r_t^n \end{array}$$

for $n = 0, 1, 2, \ldots,$ where the $\propto_{i,j}$ are constants

for $1 \leq i \leq t$ and $0 \leq j \leq m^i - 1$

Multiple Roots, Example I

If a linear homogeneous recurrence relation has a characteristic equation with roots 2, 2, 2, 5, 5, and 9, then the form of a general solution is:

$$
\begin{aligned}
a_n &= (\propto_{1,0} + \propto_{1,1} n + \propto_{1,2} n^2)2^n \\
&+ (\propto_{2,0} + \propto_{2,1} n)5^n \\
&+ (\propto_{3,0})9^n
\end{aligned}
$$

Multiple Roots, Example II

$a_0 = 1$, $a_1 = -2$, $a_2 = -1$, **and** $a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3}$

We see that $c_1 = -3$, $c_2 = -3$, and $c_3 = -1$

Characteristic Equation: $r^3 + 3r^2 + 3r + 1 = 0$

Since $r^3 + 3r^2 + 3r + 1 = (r+1)^3$, the characteristic equation has a single root, $r = -1$, of multiplicity three.

By Theorem 4., the solutions of this recurrence relation are of the form:

$$a_n = \propto_{1,0} (-1)^n + \propto_{1,1} n(-1)^n + \propto_{1,2} n^2(-1)^n$$

for some constants $\propto_{1,0}$, $\propto_{1,1}$, and $\propto_{1,2}$

Multiple Roots, Example II — Cont.

From the initial conditions, it follows that:

$$\begin{array}{rcrcl} a_0 & = & 1 & = & \propto_{1,0} (-1)^0 + \propto_{1,1} 0^1(-1)^0 + \propto_{1,2} 0^2(-1)^0 \\ a_1 & = & \text{-2} & = & \propto_{1,0} (-1)^1 + \propto_{1,1} 1^1(-1)^1 + \propto_{1,2} 1^2(-1)^1 \\ a_2 & = & \text{-1} & = & \propto_{1,0} (-1)^2 + \propto_{1,1} 2^1(-1)^2 + \propto_{1,2} 2^2(-1)^2 \end{array}$$

or

$$\begin{array}{rcl} 1 & = & \propto_{1,0} \\ \text{-2} & = & -\propto_{1,0} - \propto_{1,1} - \propto_{1,2} \\ \text{-1} & = & \propto_{1,0} + 2\propto_{1,1} + 4\propto_{1,2} \end{array}$$

Multiple Roots, Example II — Cont.

Solving these three equations simultaneously yields:

$$\propto_{1,0} = 1, \qquad \propto_{1,1} = 3, \qquad \propto_{1,2} = -2$$

Thus, the unique solution to the recurrence relation and initial conditions is the sequence $\{a_n\}$ with:

$$a_n = (1 + 3n - 2n^2)(-1)^n$$

Multiple Roots, Example III

$$a_0 = 1, \quad a_1 = 1, \quad a_2 = 2, \quad \textbf{and} \quad a_n = 3a_{n-1} - 3a_{n-2} + a_{n-3}$$

We see that $c_1 = 3$, $c_2 = -3$, and $c_3 = 1$

Characteristic Equation: $r^3 - 3r^2 + 3r - 1 = 0$

Since $r^3 - 3r^2 + 3r - 1 = (r-1)^3$, the characteristic equation has a single root, $r = 1$, of multiplicity three.

By Theorem 4., the solutions of this recurrence relation are of the form:

$$a_n = \propto_{1,0} (1)^n + \propto_{1,1} n(1)^n + \propto_{1,2} n^2(1)^n$$

for some constants $\propto_{1,0}$, $\propto_{1,1}$, and $\propto_{1,2}$

Multiple Roots, Example III — Cont.

From the initial conditions, it follows that:

$$\begin{array}{lllllllll} a_0 & = & 1 & = & \propto_{1,0}(1)^0 & + & \propto_{1,1}0^1(1)^0 & + & \propto_{1,2}0^2(1)^0 \\ a_1 & = & 1 & = & \propto_{1,0}(1)^1 & + & \propto_{1,1}1^1(1)^1 & + & \propto_{1,2}1^2(1)^1 \\ a_2 & = & 2 & = & \propto_{1,0}(1)^2 & + & \propto_{1,1}2^1(1)^2 & + & \propto_{1,2}2^2(1)^2 \end{array}$$

or

$$\begin{array}{lllllll} 1 & = & \propto_{1,0} & & & & \\ 1 & = & \propto_{1,0} & + & \propto_{1,1} & + & \propto_{1,2} \\ 2 & = & \propto_{1,0} & + & 2\propto_{1,1} & + & 4\propto_{1,2} \end{array}$$

Solving these three equations simultaneously yields:

$$\propto_{1,0} = 1, \qquad \propto_{1,1} = -\frac{1}{2}, \qquad \propto_{1,2} = \frac{1}{2}$$

Multiple Roots, Example III — Cont.

Thus, the unique solution to the recurrence relation and initial conditions is the sequence $\{a_n\}$ with:

$$\begin{aligned} a_n &= (1 - \tfrac{1}{2}n + \tfrac{1}{2}n^2)(1)^n \\ &= 1 - \tfrac{1}{2}n + \tfrac{1}{2}n^2 \\ &= \frac{2 - n + n^2}{2} \end{aligned}$$

Multiple Roots, Example IV

$a_0 = 0$, $a_1 = 1$, $a_2 = 2$, $a_3 = 3$, **and** $a_n = 2a_{n-2} - a_{n-4}$

We see that $c_1 = 0$, $c_2 = 2$, $c_3 = 0$, and $c_4 = -1$

Characteristic Equation: $r^4 - 0r^3 - 2r^2 - 0r + 1 = 0$

or, $r^4 - 2r^2 + 1 = 0$

Since $r^4 - 2r^2 + 1 = (r^2 - 1)^2 = (r-1)^2(r+1)^2$, the characteristic equation has two roots, $r_1 = 1$ and $r_2 = -1$, each of multiplicity two.

Solutions of this recurrence relation are of the form:

$$\begin{aligned} a_n \quad = \quad & (\propto_{1,0} + \propto_{1,1}\ n)(1)^n + \\ & (\propto_{2,0} + \propto_{2,1}\ n)(-1)^n \end{aligned}$$

for some constants $\propto_{1,0}$, $\propto_{1,1}$, $\propto_{2,0}$, and $\propto_{2,1}$

Multiple Roots, Example IV — Cont.

From the initial conditions, it follows that:

$$
\begin{aligned}
a_0 &= 0 = (\propto_{1,0} + \propto_{1,1}\, 0^1)(1)^0 + (\propto_{2,0} + \propto_{2,1}\, 0^1)(-1)^0 \\
&= \propto_{1,0} + \propto_{2,0} \\
a_1 &= 1 = (\propto_{1,0} + \propto_{1,1}\, 1^1)(1)^1 + (\propto_{2,0} + \propto_{2,1}\, 1^1)(-1)^1 \\
&= \propto_{1,0} + \propto_{1,1} - \propto_{2,0} - \propto_{2,1} \\
a_2 &= 2 = (\propto_{1,0} + \propto_{1,1}\, 2^1)(1)^2 + (\propto_{2,0} + \propto_{2,1}\, 2^1)(-1)^2 \\
&= \propto_{1,0} + 2\propto_{1,1} + \propto_{2,0} + 2\propto_{2,1} \\
a_3 &= 3 = (\propto_{1,0} + \propto_{1,1}\, 3^1)(1)^3 + (\propto_{2,0} + \propto_{2,1}\, 3^1)(-1)^3 \\
&= \propto_{1,0} + 3\propto_{1,1} - \propto_{2,0} - 3\propto_{2,1}
\end{aligned}
$$

Solving these three equations simultaneously yields:

$$\propto_{1,0} = \propto_{2,0} = \propto_{2,1} = 0 \quad \text{and} \quad \propto_{1,1} = 1$$

Multiple Roots, Example IV — Cont.

Thus, the unique solution to the recurrence relation and initial conditions is the sequence $\{a_n\}$ with:

$$
\begin{aligned}
a_n &= (0 + 1n)1^n + (0 + 0n)(-1)^n \\
&= n
\end{aligned}
$$

Linear Nonhomogeneous Recurrence Relations with Constant Coefficients

LNRRwCC have the form:

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k} + F(n)$$

where $c_1, c_2, \ldots, c_k$ are real numbers, and $F(n)$ is not identically 0, depending only on n.

The **associated homogeneous recurrence relation** is:

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$$

LNRRwCC Examples

- $a_n = a_{n-1} + 2^n$
- $a_n = a_{n-1} + a_{n-2} + n^2$
- $a_n = 3a_{n-1} + n3^n$
- $a_n = a_{n-1} + a_{n-2} + a_{n-3} + n!$

Theorem 5. If $\{a_n^{(p)}\}$ is a particular solution of the nonhomogeneous linear recurrence relation with constant coefficients

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k} + F(n)$$

then every solution is of the form $\{a_n^{(p)} + a_n^{(h)}\}$, where $\{a_n^{(h)}\}$ is a solution of the associated homogeneous recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$$

Key idea: every solution to a linear nonhomogeneous recurrence relation with constant coefficients is the sum of a particular solution and a solution of the associated linear homogeneous recurrence relation.

In other words, find a particular solution... every solution is a sum of this solution and a solution of the associated linear homogeneous recurrence relation.

LNRRwCC Example

$$a_n = 3a_{n-1} + 2n$$

Associated linear homogeneous equation: $a_n = 3a_{n-1}$

with solution $\{a_n^{(h)}\} = \propto 3^n$, where $\propto$ is a constant.

Finding a Particular Solution:

Since $F(n) = 2n$ is a polynomial in n of degree 1, a good guess would be a linear function in n, $p_n = cn + d$, where c and d are constants.

So, for $a_n = 3a_{n-1} + 2n$ try:

$$\begin{aligned} cn + d &= 3(c(n-1) + d) + 2n \\ cn + d &= 3cn - 3c + 3d + 2n \\ 2cn - 3c + 2d + 2n &= 0 \\ (2+2c)n + (2d-3c) &= 0 \end{aligned}$$

Then $cn + d$ is a solution IFF

$$\begin{aligned} 2 + 2c &= 0 \\ 2c &= -2 \\ c &= -1 \end{aligned}$$

and

$$\begin{aligned} 2d - 3c &= 0 \\ 2d - 3(-1) &= 0 \\ 2d + 3 &= 0 \\ d &= -\tfrac{3}{2} \end{aligned}$$

Thus, $a_n^{(p)} = cn + d = -n - \frac{3}{2}$ is a particular solution.

By Theorem 5, all solutions are of the form:

$$a_n = a_n^{(p)} + a_n^{(h)} = -n - \tfrac{3}{2} + \propto 3^n$$

1) When $a_1 = 3$, then $n = 1$ and:

$$\begin{aligned} a_1 = 3 &= -1 - \tfrac{3}{2} + \propto 3^1 \\ 3 &= -\tfrac{5}{2} + \propto 3^1 \\ \tfrac{11}{2} &= \propto 3^1 \\ \tfrac{11}{6} &= \propto \end{aligned}$$

So the solution is: $a_n = -n - \frac{3}{2} + (\frac{11}{6})3^n$

To double check, determine a_2 using both the original definition and the new closed form:

$$
\begin{aligned}
a_2 &= 3a_1 + 2(2) \\
&= 9 + 4 \\
&= 13
\end{aligned}
$$

and

$$
\begin{aligned}
a_2 &= -n - \tfrac{3}{2} + \tfrac{11}{6}3^n \\
&= -2 - \tfrac{3}{2} + \tfrac{11}{6}(9) \\
&= -\tfrac{7}{2} + \tfrac{11*3}{2} \\
&= \tfrac{33-7}{2} = \tfrac{26}{2} \\
&= 13 \quad \surd
\end{aligned}
$$

ii) Now, suppose $a_1 = 2$. Then $n = 1$, and:

$$\begin{aligned} a_1 = 2 &= -1 - \tfrac{3}{2} + \propto 3^1 \\ 2 &= -\tfrac{5}{2} + \propto 3^1 \\ \tfrac{9}{2} &= \propto 3^1 \\ \tfrac{3}{2} &= \propto \end{aligned}$$

So the solution is: $a_n = -n - \frac{3}{2} + (\frac{3}{2})3^n$

To double check, determine a_2 using both the original definition and the new closed form:

$$
\begin{aligned}
a_2 &= 3a_1 \ + \ 2(2) \\
&= 3(2) \ + \ 4 \\
&= 10
\end{aligned}
$$

and

$$
\begin{aligned}
a_2 &= -n - \frac{3}{2} + \frac{3}{2}3^n \\
&= -2 - \frac{3}{2} + \frac{3}{2}(9) \\
&= -\frac{7}{2} + \frac{27}{2} \\
&= \frac{20}{2} \\
&= 10 \quad \surd
\end{aligned}
$$