

Section 1.4 Sets

A *set* is a collection or group of objects or *elements* or *members*. (Cantor 1895)

- A set is said to *contain* its elements.
- There must be an underlying universal set U , either specifically stated or understood.

Notation:

- list the elements between braces:

$$S = \{a, b, c, d\} = \{b, c, a, d, d\}$$

(Note: listing an object more than once does not change the set. Ordering means nothing.)

- specification by predicates:

$$S = \{x \mid P(x)\},$$

S contains all the elements from U which make the predicate P true.

- brace notation with ellipses:

$$S = \{ \dots, -3, -2, -1 \},$$

the negative integers.

Common Universal Sets

- \mathbf{R} = reals
- \mathbf{N} = natural numbers = $\{0, 1, 2, 3, \dots\}$, the *counting* numbers
- \mathbf{Z} = all integers = $\{\dots, -3, -2, -1, 0, 1, 2, 3, 4, \dots\}$
- \mathbf{Z}^+ is the set of positive integers

Notation:

x is a member of S or x is an element of S :

$$x \in S.$$

x is not an element of S :

$$x \notin S.$$

Subsets

Definition: The set A is a *subset* of the set B , denoted $A \subseteq B$, iff

$$\forall x [x \in A \implies x \in B]$$

Definition: The *void* set, the *null* set, the *empty* set, denoted \emptyset , is the set with no members.

Note: the assertion $\forall x [x \in \emptyset]$ is always false. Hence

$$\forall x [x \in \emptyset \implies x \in B]$$

is always true (vacuously). Therefore, \emptyset is a subset of every set.

Note: A set B is always a subset of itself.

Definition: If $A \subseteq B$ but $A \neq B$ then we say A is a *proper* subset of B , denoted $A \subset B$ (in some texts).

Definition: The set of all subset of a set A , denoted $P(A)$, is called the *power set* of A .

Example: If $A = \{a, b\}$ then

$$P(A) = \{ \emptyset, \{a\}, \{b\}, \{a,b\} \}$$

Definition: The number of (distinct) elements in A , denoted $|A|$, is called the *cardinality* of A .

If the cardinality is a natural number (in \mathbb{N}), then the set is called *finite*, else *infinite*.

Example:

$$A = \{a, b\},$$

$$|\{a, b\}| = 2,$$

$$|P(\{a, b\})| = 4.$$

A is finite and so is $P(A)$.

Useful Fact: $|A|=n$ implies $|P(A)| = 2^n$

\mathbb{N} is infinite since $|\mathbb{N}|$ is not a natural number. It is called a *transfinite cardinal number*.

Note: Sets can be both members and subsets of other sets.

Example:

$$A = \{ \emptyset, \{ \emptyset \} \}.$$

A has two elements and hence four subsets:

$$\emptyset, \{ \emptyset \}, \{ \{ \emptyset \} \}, \{ \emptyset, \{ \emptyset \} \}$$

Note that $\{ \emptyset \}$ is both a member of A and a subset of A!

Russell's paradox: Let S be the set of all sets which are not members of themselves. Is S a member of itself?

Another paradox: Henry is a barber who shaves all people who do not shave themselves. Does Henry shave himself?

Definition: The *Cartesian product* of A with B, denoted $A \times B$, is the set of ordered pairs $\{ \langle a, b \rangle \mid a \in A \wedge b \in B \}$

$$\text{Notation: } \prod_{i=1}^n A_i = \{ \langle a_1, a_2, \dots, a_n \rangle \mid a_i \in A_i \}$$

Note: The Cartesian product of anything with \emptyset is \emptyset .
(why?)

Example:

$$A = \{a, b\}, B = \{1, 2, 3\}$$

$$A \times B = \{\langle a, 1 \rangle, \langle a, 2 \rangle, \langle a, 3 \rangle, \langle b, 1 \rangle, \langle b, 2 \rangle, \langle b, 3 \rangle\}$$

What is $B \times A$? $A \times B \times A$?

If $|A| = m$ and $|B| = n$, what is $|A \times B|$?
