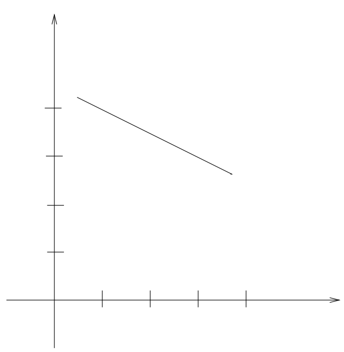
Basic Algebra Review for Math 1441

0 Introduction On your first day of Math 1441, you will take a Basic Algebra Exam. You should pass this exam in order to stay in the course. This material will help you review this elementary material in advance of the first day. If you cannot master these algebra skills, you should instead take Math 1400. These skills and more advanced skills will be heavily used in Math 1441.

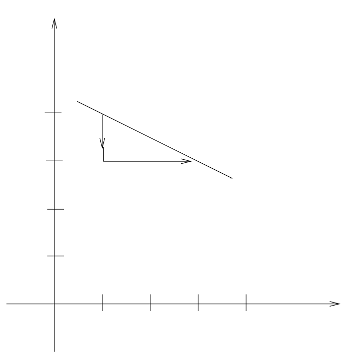
1 Slope

Recall that slope m is defined as $\frac{rise}{run}$ or $\frac{change\ in\ y}{change\ in\ x}$. Rise is the distance traveled vertically and is positive if traveling up (in the positive y direction). Run is the distance traveled horizontally and is positive if traveling right (in the positive x direction). We estimate the slope of a line by drawing horizontal and vertical lines and estimating rise and run.

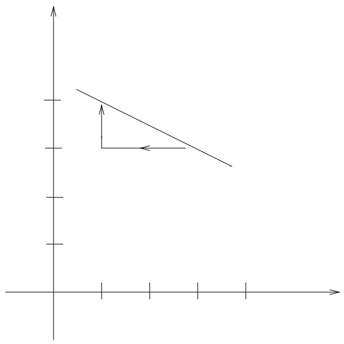
Example: Approximate the slope of the given line.



Solution: We draw in horizontal and vertical lines and estimate the run as 2 and the rise as -1. This gives an approximate slope of $-\frac{1}{2}$.

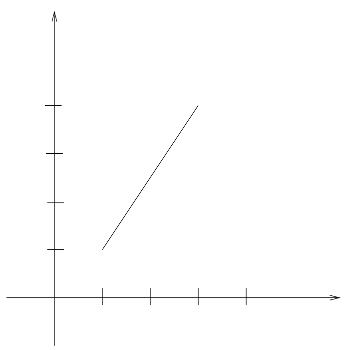


Note: If we had traveled in the other direction, we would estimate the run to be -2 and the rise to be 1, giving the same answer of $-\frac{1}{2}$.

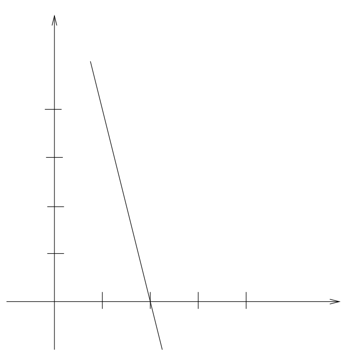


Try the following problems. The answers are in the solution section.

1. Approximate the slope of the given line:



2. Approximate the slope of the given line:



2 Lines

A line which goes through points (x_1, y_1) and (x_2, y_2) has slope $m = \frac{\textit{change in } y}{\textit{change in } x} = \frac{y_2 - y_1}{x_2 - x_1}$. If a line has slope m and goes through point (x_1, y_1), we call an arbitrary point on the line (x, y) and get that $m = \frac{y - y_1}{x - x_1}$. Rewriting, we get the usual form $y - y_1 = m(x - x_1)$.

Example: Find the line through points $(2, -3)$ and $(1, 4)$:
The slope is $m = \frac{4 - -3}{1 - 2} = \frac{7}{-1} = -7$. To find the equation, we can use either point. Using the point $(2, -3)$, we get $y - -3 = -7(x - 2)$.

The equation of the line with slope m and y intercept b is $y - b = m(x - 0)$ or $y = mx + b$. (Recall: Since points on the y axis have $x = 0$, a y intercept of b indicates that point $(0, b)$ is on the line.)

Try the following problems. Recall that parallel lines have the same slopes. The answers are in the solutions section.

1. Find the equation of the line through points $(4, -2)$ and $(-1, 3)$.
2. Find the equation of the line with x intercept 3 and y intercept -2.
3. Find the equation of the line that goes through the point $(2, 3)$ and is parallel to the line $2x + 6y = 1$.

3 Exponents

When you get confused about the rules for exponents, think about what the exponent means.

Example: $x^4x^2 = (xxxx)(xx) = xxxxxx = x^6$. The exponents are added.

Example: $(x^4)^2 = (x^4)(x^4) = (xxxx)(xxxx) = xxxxxxxx = x^8$. The exponents are multiplied.

Example: $\frac{x^5}{x^2} = \frac{xxxxx}{xx} = \left(\frac{x}{x}\right)\left(\frac{x}{x}\right)xxx = (1)(1)xxx = x^3$. The exponents are subtracted.

Example:$(xy)^2 = (xy)(xy) = xxyy = x^2y^2$. In general, $(xy)^n = x^ny^n$.

Beware: $(x+y)^2 \neq x^2+y^2$ since $(x+y)^2 = (x+y)(x+y) = x^2+xy+xy+y^2 = x^2+2xy+y^2$

Summary of Rules of Exponents:

1. $x^nx^m = x^{n+m}$
2. $(x^n)^m = x^{nm}$
3. $\frac{x^n}{x^m} = x^{n-m}$
4. $(xy)^n = x^ny^n$

Example: $\dfrac{3x^4(xy)^2}{(2x^2)^3} = \dfrac{3x^4x^2y^2}{2^3(x^2)^3} = \dfrac{3x^6y^2}{8x^6} = \dfrac{3x^{6-6}y^2}{8} = \dfrac{3y^2}{8}$

Try the following problems, simplifying completely and writing answers with no negative exponents. The answers are in the solution section.

1. $\dfrac{(3x^2y^3)^2x^3}{4xy^7}$

2. $\left(\dfrac{2x}{y^2}\right)^3(xy^3)^2$

4 Negative and Fractional Exponents

We will use the rules of exponents in the above section to understand what negative and fractional exponents mean.

1. $x^0 = 1$ Why? $\dfrac{x^2}{x^2} = 1$ and $\dfrac{x^2}{x^2} = x^{2-2} = x^0$.

2. $x^{-n} = \dfrac{1}{x^n}$ Why? Since $x^nx^{-n} = x^{n+-n} = x^0 = 1$, we know

   $x^nx^{-n} = 1$. When we divide both sides by x^n, we get $x^{-n} = \dfrac{1}{x^n}$.

3. $x^{\frac{1}{n}} = \sqrt[n]{x}$ Why? $(x^{\frac{1}{n}})^n = x^{\frac{n}{n}} = x^1 = x$. So $(x^{\frac{1}{n}})^n = x$ and we

   take the nth root of both sides to get $x^{\frac{1}{n}} = \sqrt[n]{x}$.

4. $x^{\frac{m}{n}} = \sqrt[n]{x^m} = (\sqrt[n]{x})^m$ Why? $x^{\frac{m}{n}} = (x^m)^{\frac{1}{n}} = \sqrt[n]{x^m}$ and

   $x^{\frac{m}{n}} = (x^{\frac{1}{n}})^m = (\sqrt[n]{x})^m$.

Example: Write $x^{-\frac{2}{3}}$ without fractions or negatives in the exponent: $x^{-\frac{2}{3}} = \dfrac{1}{x^{\frac{2}{3}}} = \dfrac{1}{\sqrt[3]{x^2}}$.
Another correct answer would be $\dfrac{1}{(\sqrt[3]{x})^2}$.

Example: Write $\dfrac{1}{x^2}$ in the form x^a: $\dfrac{1}{x^2} = x^{-2}$

Example: Write $\frac{1}{\sqrt[5]{x^3}}$ in the form x^a: $\frac{1}{\sqrt[5]{x^3}} = \frac{1}{(x^3)^{\frac{1}{5}}} = \frac{1}{x^{\frac{3}{5}}} = x^{-\frac{3}{5}}$.

Try the following problems. The answers are in the solution section.

1. Write $x^{\frac{3}{4}}$ without fractions or negatives in the exponent.
2. Write $x^{-\frac{3}{2}}$ without fractions or negatives in the exponent.
3. Write $(\sqrt{x})^5$ in the form x^a.
4. Write $\frac{1}{\sqrt[3]{x^7}}$ in the form x^a.

5 Fractions

We'll look at some numerical examples first. For each example, try it yourself before you look at the solution.

Example: $\frac{2}{5}\ \frac{3}{7}$ Answer: $\frac{(2)(3)}{(5)(7)} = \frac{6}{35}$

Example: $\frac{\frac{2}{5}}{\frac{3}{7}}$ Answer: $\frac{2}{5}\ \frac{7}{3} = \frac{14}{15}$

Example: $\frac{2}{5} + \frac{3}{7}$ $Answer: \frac{2}{5}\ \frac{7}{7} + \frac{3}{7}\ \frac{5}{5} = \frac{14}{35} + \frac{15}{35} = \frac{29}{35}$

Now let's try some examples with variables. Try yourself before looking at answers.

Example: $\frac{x}{y}\ z$ Answer: $\frac{x}{y}\ \frac{z}{1} = \frac{xz}{y}$

Example: $\frac{\frac{x}{y}}{\frac{z}{y}}$ Answer: $\frac{x}{y}\frac{y}{z} = \frac{xy}{yz} = \frac{x}{z}$

Example: $\frac{x}{y} - \frac{y}{z}$ Answer: $\frac{x}{y}\ \frac{z}{z} - \frac{y}{z}\ \frac{y}{y} = \frac{xz - y^2}{yz}$

Example: $\dfrac{2}{x+1} - \dfrac{3}{x-2}$ Answer: $\dfrac{2}{x+1}\,\dfrac{x-2}{x-2} - \dfrac{3}{x-2}\,\dfrac{x+1}{x+1} = \dfrac{2(x-2)-3(x+1)}{(x+1)(x-2)}$

$$= \frac{2x-4-3x-3}{(x+1)(x-2)} = \frac{-x-7}{x^2-x-2}$$

Try the following problems. The answers are in the solution section.

1. $\dfrac{\frac{x^2-x}{y}}{\frac{x}{y}}$
2. $\dfrac{x+1}{x-2} - \dfrac{x+3}{x-4}$

6 Complex Fractions

The first step in simplifying complex fractions is to write the numerator as one fraction and the denominator as one fraction.

Example: $\dfrac{\frac{x}{y} - \frac{y}{z}}{\frac{x}{z} - \frac{z}{y}} = \dfrac{\frac{xz-y^2}{yz}}{\frac{xy-z^2}{yz}} = \dfrac{xz-y^2}{yz}\,\dfrac{yz}{xy-z^2} = \dfrac{xz-y^2}{xy-z^2}$

Example: $\dfrac{2+\frac{1}{x}}{x} = \dfrac{\frac{2x}{x}+\frac{1}{x}}{\frac{x}{1}} = \dfrac{\frac{2x+1}{x}}{\frac{x}{1}} = \dfrac{2x+1}{x}\,\dfrac{1}{x} = \dfrac{2x+1}{x^2}$

Here are some to try. Answers are in the solution section.

1. $\dfrac{x+\frac{1}{3}}{3+\frac{1}{x}}$
2. $\dfrac{3-\frac{1}{x+1}}{\frac{3}{x+1}-x}$

7 Functions

For the function defined by $f(x) = 3x - x^2$, we express in English that the function takes a number, multiplies it by 3 and subtracts its square.

So $f(-2) = 3(-2) - (-2)^2 = -6 - 4 = -10$ and $f(x + h) = 3(x + h) - (x + h)^2 = 3x + 3h - (x^2 + 2xh + h^2) = 3x + 3h - x^2 - 2xh - h^2$.

For the function defined by $g(x) = 1 - 3x^2$, we express in English that the function takes a number, squares it, multiplies the result by 3 and subtracts that result from 1.

So $g(-2) = 1 - 3(-2)^2 = 1 - 3(4) = 1 - 12 = -11$ and $g(x + h) = 1 - 3(x + h)^2 = 1 - 3(x^2 + 2xh + h^2) = 1 - 3x^2 - 6xh - 3h^2$.

Here are some for you to try. The answers are in the solution section.

1. If $f(x) = 2x - x^2$, find $f(x + h)$ and simplify completely.
2. If $f(x) = \dfrac{x+1}{2x-1}$, find $f(x + h)$ and simplify completely.
3. If $f(x) = 3 - 2x^2$, find $f(x + h)$ and simplify completely.

8 Linear Equations

Rewrite the equation so that there are no variables in the denominator and no parentheses. Move all terms with the variable for which we are solving on one side of the equation and move all terms without that variable on the other side. Factor out the variable and solve.

Example: Solve $3(x + 2) = 4x + 1$ for x: We distribute to get $3x + 6 = 4x + 1$. Isolating the x's on one side, we get $3x - 4x = 1 - 6$ or $-x = -5$. So the answer is $x = 5$.

Example: Solve $\dfrac{y}{2x} = y + 1$ for y: Multiply both sides by $2x$ to get $y = 2x(y + 1)$ or $y = 2xy + 2x$. Isolate the terms with y to get $y - 2xy = 2x$ or $y(1 - 2x) = 2x$. The solution is $y = \dfrac{2x}{1-2x}$.

Try the following problems. The answers are in the solution section.

1. Solve $\dfrac{1}{A} + \dfrac{1}{B} = \dfrac{1}{C}$ for B.
2. Solve $A(B + C) = BC + A$ for C.

9 Quadratic Equations

We can solve quadratic equations by factoring or by using the quadratic equation.

Example: Solve $x^2 - x = 6$ by factoring: We put all terms on one side, getting $x^2 - x - 6 = 0$. Next, factor to get $(x-3)(x+2) = 0$. If a product ab equals 0, either $a = 0$ or $b = 0$. For our problem, we conclude $x - 3 = 0$ or $x + 2 = 0$. The answer is $x = 3$ or $x = -2$.

Notice that, if $ab = 6$, we cannot conclude $a = 6$ or $b = 6$. Maybe $a = 2$ and $b = 3$ or $a = \frac{1}{2}$ and $b = 12$. So, if you factored without moving all terms to one side to get $x(x-1) = 6$, you cannot conclude $x = 6$ or $x - 1 = 6$.

Example: Solve $x^2 - x = 6$ by using the quadratic formula: Again, we need to put all terms on one side, getting $x^2 - x - 6 = 0$. To solve $ax^2 + bx + c = 0$, the quadratic formula tells us that

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

In this example, $a = 1$, $b = -1$, and $c = -6$, so $x = \frac{- -1 \pm \sqrt{(-1)^2 - 4(1)(-6)}}{(2)(1)} = \frac{1 \pm \sqrt{25}}{2} = \frac{1 \pm 5}{2}$. The solution is $x = \frac{1+5}{2} = 3$ or $x = \frac{1-5}{2} = -2$.

Here are some problems to try. The answers are in the solution section.

1. Solve the equation for x: $x^2 - 2x = 8$
2. Solve the equation for x: $\frac{2}{x} + 1 = x + 2$

10 Common Mistakes

Remember that $\sqrt{x^2 + y^2} \neq x + y$. For example, $\sqrt{2^2 + 3^2} \neq 2 + 3$ since $\sqrt{2^2 + 3^2} = \sqrt{13}$. Similarly, $\sqrt{x^2 - y^2} \neq x - y$. However, if x and y are nonnegative, $\sqrt{x^2 y^2} = xy$ and $\sqrt{\frac{x^2}{y^2}} = \frac{x}{y}$.

In a similar vein, $(x+y)^2 \neq x^2 + y^2$ since $(x+y)^2 = (x+y)(x+y) = x(x+y) + y(x+y) = x^2 + xy + yx + y^2 = x^2 + 2xy + y^2$. However, $(xy)^2 = x^2 y^2$ since $(xy)^2 = (xy)(xy) = xxyy = x^2 y^2$. Also, $(x-y)^2 \neq x^2 - y^2$ but $\left(\frac{x}{y}\right)^2 = \frac{x^2}{y^2}$.

Care needs to be taken when deciding if canceling is possible. We have that $\frac{x+y}{x} \neq y$ but $\frac{xy}{x} = y$ since $\frac{xy}{x} = \left(\frac{x}{x}\right)y = (1)y = y$.

Recall that, for inequalities, if we multiply or divide both sides by a negative number, the inequality sign gets changed. We know that $-3 \leq 2$ but, if we multiply by -2, we get $(-3)(-2) \geq (2)(-2)$ since $6 \geq -4$.

Example: Solve $3 - 2x \leq 9$ for x: We get $-2x \leq 9 - 3$ or $-2x \leq 6$ or $x \geq -3$. The answer in interval notation is $[-3, \infty)$.

Example: Solve $x + 2 > 4x - 1$ for x: We get $x - 4x > -1 - 2$ or $-3x > -3$ or $x < 1$. The answer in interval notation is $(-\infty, 1)$.

Decide if the following statements are true or false. Assume that all variable are nonnegative. Answers are in the solution section.

1. $\sqrt{16+9} = 4 + 3 = 7$
2. If $3x > 12$, then $x > 4$.
3. $\frac{x+3y}{y} = x + 3$
4. $(2+3)^3 = 8 + 27$
5. If $-2x < 4$, $x \geq -2$.
6. $\sqrt[3]{\frac{x^3}{y^3}} = \frac{x}{y}$
7. If $\frac{x}{-4} > 3$, then $x < -12$.
8. $\sqrt{x^2 - 9} = x - 3$
9. $\sqrt{9x^2} = 3x$
10. $\frac{xy+3z}{z} = xy + 3$
11. $(3x)^2 = 3x^2$
12. $\frac{xy^2z}{yz} = xy$
13. $(2y)^3 = 8y^3$

11 Solutions

Section 1:

1. $m = \dfrac{rise}{run} = \dfrac{3}{2}$
2. $m = \dfrac{rise}{run} = \dfrac{-4}{1} = \dfrac{4}{-1} = -4$

Section 2:

1. $m = \dfrac{3 - -2}{-1-4} = \dfrac{5}{-5} = -1$ and the equation of the line is $y - (-2) = -1(x-4)$.
2. The points are $(3,0)$ and $(0,-2)$. So $m = \dfrac{-2-0}{0-3} = \dfrac{-2}{-3} = \dfrac{2}{3}$ and the equation of the line is $y - 0 = \dfrac{2}{3}(x-3)$.
3. To find the slope of the line $2x + 6y = 1$, we solve for y: $6y = 1 - 2x$ so $y = \dfrac{1}{6} - \dfrac{2}{6}x$. The slope is $-\dfrac{2}{6} = -\dfrac{1}{3}$ and the equation of the parallel line is $y - 3 = -\dfrac{1}{3}(x-2)$.

Section 3:

1. $\dfrac{(3x^2y^3)^2x^3}{4xy^7} = \dfrac{3^2(x^2)^2(y^3)^2x^3}{4xy^7} = \dfrac{9x^4y^6x^3}{4xy^7} = \dfrac{9x^7y^6}{4xy^7} = \dfrac{9x^{7-1}y^{6-7}}{4} = \dfrac{9x^6y^{-1}}{4} = \dfrac{9x^6}{4y}$
2. $\left(\dfrac{2x}{y^2}\right)^3(xy^3)^2 = \dfrac{(2)^3x^3}{(y^2)^3}(x)^2(y^3)^2 = \dfrac{8x^3}{y^6}\ \dfrac{x^2y^6}{1} = \dfrac{8x^5y^6}{y^6} = 8x^5$

Section 4:

1. $x^{\frac{3}{4}} = (x^3)^{\frac{1}{4}} = \sqrt[4]{x^3}$. Also $x^{\frac{3}{4}} = (x^{\frac{1}{4}})^3 = (\sqrt[4]{x})^3$.
2. $x^{-\frac{3}{2}} = \dfrac{1}{x^{\frac{3}{2}}} = \dfrac{1}{\sqrt{x^3}}$
3. $(\sqrt{x})^5 = (x^{\frac{1}{2}})^5 = x^{\frac{5}{2}}$

4. $\dfrac{1}{\sqrt[3]{x^7}} = \dfrac{1}{(x^7)^{\frac{1}{3}}} = \dfrac{1}{x^{\frac{7}{3}}} = x^{-\frac{7}{3}}$

Section 5:

1. $\dfrac{\frac{x^2-x}{y}}{\frac{x}{y}} = \dfrac{x^2-x}{y} \quad \dfrac{y}{x} = \dfrac{x^2-x}{x} = \dfrac{x(x-1)}{x} = x-1$

2. $\dfrac{x+1}{x-2} - \dfrac{x+3}{x-4} = \dfrac{x+1}{x-2}\,\dfrac{x-4}{x-4} - \dfrac{x+3}{x-4}\,\dfrac{x-2}{x-2} = \dfrac{(x+1)(x-4)}{(x-2)(x-4)} - \dfrac{(x+3)(x-2)}{(x-4)(x-2)} =$

   $\dfrac{(x+1)(x-4)-(x+3)(x-2)}{(x-2)(x-4)} = \dfrac{x^2-3x-4-(x^2+x-6)}{x^2-6x+8} = \dfrac{-4x+2}{x^2-6x+8}$

Section 6:

1. $\dfrac{x+\frac{1}{3}}{3+\frac{1}{x}} = \dfrac{\frac{3x}{3}+\frac{1}{3}}{\frac{3x}{x}+\frac{1}{x}} = \dfrac{\frac{3x+1}{3}}{\frac{3x+1}{x}} = \dfrac{3x+1}{3}\,\dfrac{x}{3x+1} = \dfrac{x}{3}$

2. $\dfrac{3-\frac{1}{x+1}}{\frac{3}{x+1}-x} = \dfrac{\frac{3(x+1)-1}{x+1}}{\frac{3-x(x+1)}{x+1}} = \dfrac{\frac{3x+2}{x+1}}{\frac{3-x^2-x}{x+1}} = \dfrac{3x+2}{x+1} \quad \dfrac{x+1}{3-x^2-x} = \dfrac{3x+2}{3-x^2-x}$

Section 7:

1. $f(x+h) = 2(x+h)-(x+h)^2 = 2x+2h-(x^2+2xh+h^2) = 2x+2h-x^2-2xh-h^2$

2. $f(x+h) = \dfrac{x+h+1}{2(x+h)-1} = \dfrac{x+h+1}{2x+2h-1}$

3. $f(x+h) = 3-2(x+h)^2 = 3-2(x^2+2xh+h^2) = 3-2x^2-4xh-2h^2$

Section 8:

1. Multiplying the equation $\dfrac{1}{A}+\dfrac{1}{B}=\dfrac{1}{C}$ by ABC on both sides, we get $BC+AC=AB$. We isolate the terms with B on the left to get $BC-AB=-AC$. So $B(C-A)=-AC$ and $B=\dfrac{-AC}{C-A}$

2. Distributing, we get $AB+AC=BC+A$. We isolate the terms with C on the left and get $AC-BC=A-AB$. So $C(A-B)=A-AB$ and $C=\dfrac{A-AB}{A-B}$.

Section 9:

1. $x^2 - 2x - 8 = 0$. Factoring, we get $(x-4)(x+2) = 0$ so $x - 4 = 0$ or $x + 2 = 0$. The answers are $x = 4, -2$.
2. Multiplying both sides of the equation by x, we get $2 + x = x^2 + 2x$. So $0 = x^2 + x - 2$ and $0 = (x+2)(x-1)$. The answer is $x = -2, 1$.

Section 10:

1. False: $\sqrt{16+9} = \sqrt{25} = 5$
2. True
3. False
4. False: $(2+3)^3 = 5^3 = 125$
5. False: If $-2x < 4$ then $x > -2$.
6. True
7. True
8. False
9. True
10. False
11. False: $(3x)^2 = 9x^2$
12. True
13. True