Basic Algebra Review for Math 1400

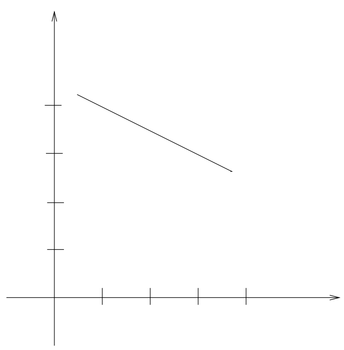
0 Introduction

On your first day of Math 1400, you will take a Basic Algebra Exam. You should pass this exam in order to stay in the course. This material will help you review this elementary material in advance of the first day. If you cannot master these algebra skills, you should instead take Math 1271. These skills and more advanced skills will be heavily used in Math 1400.

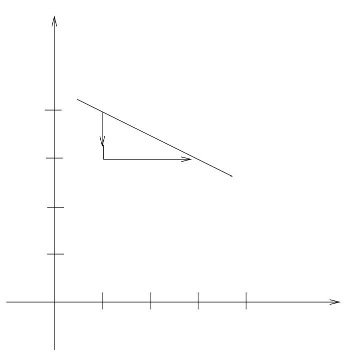
1 Slope

Recall that slope m is defined as $\frac{rise}{run}$ or $\frac{change\ in\ y}{change\ in\ x}$. Rise is the distance traveled vertically and is positive if traveling up (in the positive y direction). Run is the distance traveled horizontally and is positive if traveling right (in the positive x direction). We estimate the slope of a line by drawing horizontal and vertical lines and estimating rise and run.

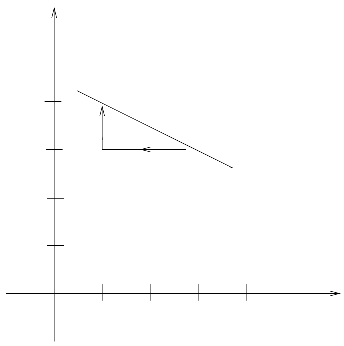
Example: Approximate the slope of the given line.



Solution: We draw in horizontal and vertical lines and estimate the run as 2 and the rise as -1. This gives an approximate slope of $-\frac{1}{2}$.

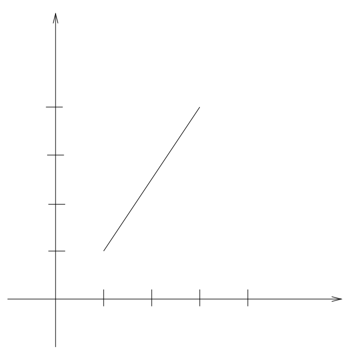


Note: If we had traveled in the other direction, we would estimate the run to be -2 and the rise to be 1, giving the same answer of $-\frac{1}{2}$.

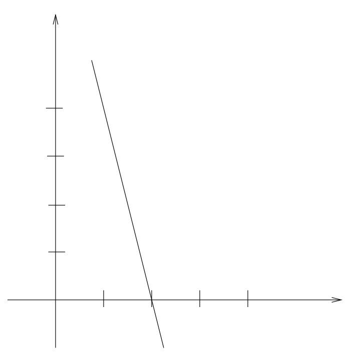


Try the following problems. The answers are in the solution section.

1. Approximate the slope of the given line:



2. Approximate the slope of the given line:



2 Lines

A line which goes through points (x_1, y_1) and (x_2, y_2) has slope $m = \frac{\textit{change in } y}{\textit{change in } x} = \frac{y_2 - y_1}{x_2 - x_1}$. If a line has slope m and goes through point (x_1, y_1), we call an arbitrary point on the line (x, y) and get that $m = \frac{y - y_1}{x - x_1}$. Rewriting, we get the usual form $y - y_1 = m(x - x_1)$.

Example: Find an equation of the line through points $(2, -3)$ and $(1, 4)$:
The slope is $m = \frac{4 - -3}{1 - 2} = \frac{7}{-1} = -7$. To find an equation, we can use either point. Using the point $(2, -3)$, we get $y - -3 = -7(x - 2)$.

The equation of the line with slope m and y intercept b is $y - b = m(x - 0)$ or $y = mx + b$. (Recall: Since points on the y axis have $x = 0$, a y intercept of b indicates that point $(0, b)$ is on the line.)

Try the following problems. The answers are in the solutions section.

1. Find an equation of the line through points $(4, -2)$ and $(-1, 3)$.
2. Find an equation of the line with x intercept 3 and y intercept -2.

3 Exponents

When you get confused about the rules for exponents, think about what the exponent means.

Example: $x^4x^2 = (xxxx)(xx) = xxxxxx = x^6$. The exponents are added.

Example: $(x^4)^2 = (x^4)(x^4) = (xxxx)(xxxx) = xxxxxxxx = x^8$. The exponents are multiplied.

Example: $\frac{x^5}{x^2} = \frac{xxxxx}{xx} = \left(\frac{x}{x}\right)\left(\frac{x}{x}\right)xxx = (1)(1)xxx = x^3$. The exponents are subtracted.

Example:$(xy)^2 = (xy)(xy) = xxyy = x^2y^2$. In general, $(xy)^n = x^ny^n$.

Beware: $(x+y)^2 \neq x^2+y^2$ since $(x+y)^2 = (x+y)(x+y) = x^2+xy+xy+y^2 = x^2+2xy+y^2$

Summary of Rules of Exponents:

1. $x^nx^m = x^{n+m}$
2. $(x^n)^m = x^{nm}$
3. $\frac{x^n}{x^m} = x^{n-m}$
4. $(xy)^n = x^ny^n$

Example: $\dfrac{3x^4(xy)^2}{(2x^2)^3} = \dfrac{3x^4x^2y^2}{2^3(x^2)^3} = \dfrac{3x^6y^2}{8x^6} = \dfrac{3x^{6-6}y^2}{8} = \dfrac{3y^2}{8}$

Try the following problems, simplifying completely and writing answers with no negative exponents. The answers are in the solution section.

1. $\dfrac{(3x^2y^3)^2x^3}{4xy^7}$
2. $\left(\dfrac{2x}{y^2}\right)^3(xy^3)^2$

4 Fractions

We'll look at some numerical examples first. For each example, try it yourself before you look at the solution.

Example: $\dfrac{2}{5}\ \dfrac{3}{7}$ Answer: $\dfrac{(2)(3)}{(5)(7)} = \dfrac{6}{35}$

Example: $\frac{\frac{2}{5}}{\frac{3}{7}}$ Answer: $\dfrac{2}{5}\ \dfrac{7}{3} = \dfrac{14}{15}$

Example: $\dfrac{2}{5} + \dfrac{3}{7}$ $Answer: \dfrac{2}{5}\ \dfrac{7}{7} + \dfrac{3}{7}\ \dfrac{5}{5} = \dfrac{14}{35} + \dfrac{15}{35} = \dfrac{29}{35}$

Now let's try some examples with variables. Try yourself before looking at answers.

Example: $\dfrac{x}{y}\ z$ Answer: $\dfrac{x}{y}\ \dfrac{z}{1} = \dfrac{xz}{y}$

Example: $\frac{\frac{x}{y}}{\frac{z}{y}}$ Answer: $\dfrac{x}{y}\dfrac{y}{z} = \dfrac{xy}{yz} = \dfrac{x}{z}$

Example: $\dfrac{x}{y} - \dfrac{y}{z}$ Answer: $\dfrac{x}{y}\ \dfrac{z}{z} - \dfrac{y}{z}\ \dfrac{y}{y} = \dfrac{xz - y^2}{yz}$

Example: $\dfrac{2}{x+1} - \dfrac{3}{x-2}$ Answer: $\dfrac{2}{x+1}\,\dfrac{x-2}{x-2} - \dfrac{3}{x-2}\,\dfrac{x+1}{x+1} = \dfrac{2(x-2)-3(x+1)}{(x+1)(x-2)}$

$$= \frac{2x-4-3x-3}{(x+1)(x-2)} = \frac{-x-7}{x^2-x-2}$$

Try the following problems, combining terms and simplifying completely. The answers are in the solution section.

1. $\dfrac{2}{x} - \dfrac{3}{y}$
2. $\frac{\frac{10}{x}}{\frac{5}{y}}$
3. $3\left(\dfrac{x}{2y}\right)$
4. $3 + \dfrac{2}{x} - \dfrac{y}{5}$
5. $-x + 3 + \dfrac{x}{x+3}$
6. $\dfrac{x+1}{x-2} - \dfrac{x+3}{x-4}$

5 Algebraic Simplifications

The distributive law states that $a(b+c) = ab + ac$. We often use the distributive law when simplifying.

Example: $9x^2 - 5x + 3x = 9x^2 - 2x$ since $-5x + 3x = x(-5+3) = x(-2) = -2x$.

Example: $(2x^2+7x+3)-(x^2+4x-1) = 2x^2+7x+3-x^2-4x+1 = x^2+3x+4$. Notice that we distributed the minus sign: $-(x^2+4x-1) = -1(x^2+4x-1) = -1(x^2)-1(4x)-1(-1) = -x^2-4x+1$.

Example: $(2x+3)(x-2) = (2x+3)(x) + (2x+3)(-2) = 2x^2+3x-4x-6 = 2x^2-x-6$

Try simplifying the following problems completely. The solutions are in the solution section.

1. $8x^2 - 4x - 2(x^2 - 2x + 3)$
2. $(3x + 4)(x - 1)$
3. $\dfrac{2 + x}{x + 4}$
4. $(2x - 3)^2$

6 Linear Equations

Rewrite the equation so that there are no variables in the denominator and no parentheses. Move all terms with the variable for which we are solving on one side of the equation and move all terms without that variable on the other side. Factor out the variable and solve.

Example: Solve $2x - 3 = 5x + 1$ for x: We add 3 to both sides to get $2x = 5x + 4$. Then we subtract $5x$ from both sides to get $-3x = 4$. The solution is $x = -\dfrac{4}{3}$

Example: Solve $3(x + 2) = 4x + 1$ for x: We distribute to get $3x + 6 = 4x + 1$. Isolating the x's on one side, we get $3x - 4x = 1 - 6$ or $-x = -5$. So the answer is $x = 5$.

Example: Solve $\dfrac{3y}{2} = 3y - 6$ for y: Multiply both sides by 2 to get $3y = 6y - 12$. Subtracting $6y$ from both sides, we get $-3y = -12$ so the solution is $y = 4$.

Solve the following linear equations for x. The solutions are in the solution section.

1. $5x - 2 = 3x + 6$
2. $2x - (x - 3) = 5$
3. $2(x - 1) = 5x + 1$
4. $\dfrac{2x + 5}{x} = 3$

7 Linear Equations With More Than One Variable

The procedure is the same as in the previous section.

Example: Solve $y + 2x = 2y + 1$ for y: We put all terms with y on the left and all terms without y on the right: $y - 2y = 1 - 2x$. We conclude that $-y = 1 - 2x$ so that the solution is $y = -1 + 2x$.

Example: Solve $2B(A + C) = AC$ for A: Distributing, we get $2AB + 2BC = AC$. We put the terms with A on the left and the terms without A on the right: $2AB - AC = -2BC$. Next, we factor out the A to get $A(2B - C) = -2BC$. The solution is $A = \dfrac{-2BC}{2B - C}$.

Example: Solve $\dfrac{1}{x} - \dfrac{2}{y} = \dfrac{1}{z}$ for y: We multiply both sides by the least common denominator xyz to get $yz - 2xz = xy$. Next we put the terms with y on the left and the terms without y on the right to get $yz - xy = 2xz$. Factoring out y, we get $y(z - x) = 2xz$. The solution is $y = \dfrac{2xz}{z - x}$.

Solve the following. The solutions are in the solutions section.

1. Solve $A(4C - B) = 2BC$ for B.
2. Solve $\dfrac{B}{2C} = A + B$ for C.
3. Solve $\dfrac{3}{x} + \dfrac{4}{y} = \dfrac{1}{z}$ for x.

8 Factoring

When factoring, we first look for factors that are common to all terms.

Example: $4x^2 - 10x + 6x^3 = 2x(2x - 5 + 3x^2)$

Example: $(x + 5)(x + 2) + (x + 5)(x - 1) = (x + 5)(x + 2 + x - 1) = (x + 5)(2x + 1)$

To factor a trinomial of the form $ax^2 + bx + c$, we write $ax^2 + bx + c = (?x+?)(?x+?)$. We need to find values to insert where the question marks are to make the equation true.

Example: Factor $3x^2+11x+6$: We write $3x^2+11x+6 = (?x+?)(?x+?)$. Since $3x^2 = (3x)(x)$, we know that $3x^2 + 11x + 6 = (3x+?)(x+?)$. We also know that the product of the two missing numbers is 6, so the possibilities are 1,6 or 2,3 or 3,2 or 6,1 or -1,-6 or -2,-3 or -3,-2 or -6,-1. If we try 1,6 we get $(3x+1)(x+6) = 3x^2+18x+x+6 = 3x^2+19x+6$, an incorrect answer. If we try 2,3 we get $(3x+2)(x+3) = 3x^2+2x+9x+6 = 3x^2+11x+6$, the correct answer. So the factorization is $(3x+2)(x+3)$. We try the different possibilities until we hit upon the correct one.

The difference of squares is easy to factor: $A^2 - B^2 = (A+B)(A-B)$.

Example: Factor $x^2 - 9$: $x^2 - 9 = (x)^2 - (3)^2 = (x+3)(x-3)$

Example: Factor $4a^3 - 81a$: We first notice a common factor of a, so we get $4a^3 - 81a = a(4a^2 - 81) = a((2a)^2 - (9)^2) = a(2a+9)(2a-9)$.

Factor the following polynomials completely. The solutions are in the solutions section

1. $9ab - 12a^2 + 6ab^2$
2. $2x^2 - x - 10$
3. $5x^2 + 29x - 6$
4. $x^3 - 16x$

9 Functions

For the function defined by $f(x) = 3x - x^2$, we express in English that the function takes a number, multiplies it by 3 and subtracts its square. So $f(4) = 3(4) - (4)^2 = 12 - 16 = -4$.

For the function defined by $g(x) = 1 - 3x^2$, we express in English that the function takes a number, squares it, multiples the result by 3 and subtracts that result from 1. So $g(-2) = 1 - 3(-2)^2 = 1 - 3(4) = -11$. Notice that $(-2)^2 = (-2)(-2) = 4$, but $-2^2 = -(2)^2 = -(2)(2) = -4$.

Calculate the following, simplifying completely. The answers are in the solution section.

1. If $f(x) = 2x - x^2$, find $f(3)$.
2. If $f(x) = 3 - 2x^2 + 4x$, find $f(-2)$.
3. If $f(x) = \dfrac{x-1}{x-2}$, find $f(-3)$.

10 Common Mistakes

Remember that $\sqrt{x^2 + y^2} \neq x + y$. For example, $\sqrt{2^2 + 3^2} \neq 2 + 3$ since $\sqrt{2^2 + 3^2} = \sqrt{13}$. Similarly, $\sqrt{x^2 - y^2} \neq x - y$. However, if x and y are nonnegative, $\sqrt{x^2y^2} = xy$ and $\sqrt{\dfrac{x^2}{y^2}} = \dfrac{x}{y}$.

In a similar vein, $(x+y)^2 \neq x^2+y^2$ since $(x+y)^2 = (x+y)(x+y) = x(x+y)+y(x+y) = x^2 + xy + yx + y^2 = x^2 + 2xy + y^2$. However, $(xy)^2 = x^2y^2$ since $(xy)^2 = (xy)(xy) = xxyy = x^2y^2$. Also, $(x - y)^2 \neq x^2 - y^2$ but $\left(\dfrac{x}{y}\right)^2 = \dfrac{x^2}{y^2}$.

Care needs to be taken when deciding if canceling is possible. We have that $\dfrac{x+y}{x} \neq y$ but $\dfrac{xy}{x} = y$ since $\dfrac{xy}{x} = \left(\dfrac{x}{x}\right)y = (1)y = y$.

Recall that, for inequalities, if we multiply or divide both sides by a negative number, the inequality sign gets changed. We know that $-3 \leq 2$ but, if we multiply by -2, we get $(-3)(-2) \geq (2)(-2)$ since $6 \geq -4$.

Example: Solve $3 - 2x \leq 9$ for x: We get $-2x \leq 9 - 3$ or $-2x \leq 6$ or $x \geq -3$. The answer in interval notation is $[-3, \infty)$.

Example: Solve $x + 2 > 4x - 1$ for x: We get $x - 4x > -1 - 2$ or $-3x > -3$ or $x < 1$. The answer in interval notation is $(-\infty, 1)$.

Decide if the following statements are true or false. Assume that all variable are nonnegative. Answers are in the solution section.

1. $\sqrt{16+9} = 4 + 3 = 7$
2. If $3x > 12$, then $x > 4$.
3. $\dfrac{x+3y}{y} = x + 3$
4. $(2+3)^3 = 8 + 27$
5. If $-2x < 4$, $x \geq -2$.
6. $\sqrt[3]{\dfrac{x^3}{y^3}} = \dfrac{x}{y}$
7. If $\dfrac{x}{-4} > 3$, then $x < -12$.
8. $\sqrt{x^2 - 9} = x - 3$
9. $\sqrt{9x^2} = 3x$
10. $\dfrac{xy+3z}{z} = xy + 3$
11. $(3x)^2 = 3x^2$
12. $\dfrac{xy^2z}{yz} = xy$
13. $(2y)^3 = 8y^3$

11 Solutions

Section 1:

1. $m = \frac{rise}{run} = \frac{3}{2}$
2. $m = \frac{rise}{run} = \frac{-4}{1} = \frac{4}{-1} = -4$

Section 2:

1. $m = \frac{3 - -2}{-1 - 4} = \frac{5}{-5} = -1$ and an equation of the line is $y - (-2) = -1(x - 4)$.
2. The points are $(3, 0)$ and $(0, -2)$. So $m = \frac{-2 - 0}{0 - 3} = \frac{-2}{-3} = \frac{2}{3}$ and an equation of the line is $y - 0 = \frac{2}{3}(x - 3)$.

Section 3:

1. $\frac{(3x^2y^3)^2x^3}{4xy^7} = \frac{3^2(x^2)^2(y^3)^2x^3}{4xy^7} = \frac{9x^4y^6x^3}{4xy^7} = \frac{9x^7y^6}{4xy^7} = \frac{9x^{7-1}y^{6-7}}{4} = \frac{9x^6y^{-1}}{4} = \frac{9x^6}{4y}$
2. $\left(\frac{2x}{y^2}\right)^3(xy^3)^2 = \frac{(2)^3x^3}{(y^2)^3}(x)^2(y^3)^2 = \frac{8x^3}{y^6}\ \frac{x^2y^6}{1} = \frac{8x^5y^6}{y^6} = 8x^5$

Section 4:

1. $\frac{2}{x} - \frac{3}{y} = \frac{2}{x}\frac{y}{y} - \frac{3}{y}\frac{x}{x} = \frac{2y - 3x}{xy}$
2. $\frac{\frac{10}{x}}{\frac{5}{y}} = \frac{10}{x}\frac{y}{5} = \frac{10y}{5x} = \frac{2y}{x}$
3. $3\left(\frac{x}{2y}\right) = \left(\frac{3}{1}\right)\left(\frac{x}{2y}\right) = \frac{3x}{2y}$
4. $3 + \frac{2}{x} - \frac{y}{5} = \left(\frac{3}{1}\right)\left(\frac{5x}{5x}\right) + \left(\frac{2}{x}\right)\left(\frac{5}{5}\right) - \left(\frac{y}{5}\right)\left(\frac{x}{x}\right) = \frac{15x + 10 - xy}{5x}$
5. $-x + 3 + \frac{x}{x+3} = \left(\frac{-x}{1}\right)\left(\frac{x+3}{x+3}\right) + \left(\frac{3}{1}\right)\left(\frac{x+3}{x+3}\right) + \frac{x}{x+3} = \frac{-x^2 - 3x + 3x + 9 + x}{x+3} = \frac{-x^2 + x + 9}{x+3}$

6. $\dfrac{x+1}{x-2} - \dfrac{x+3}{x-4} = \left(\dfrac{x+1}{x-2}\right)\left(\dfrac{x-4}{x-4}\right) - \left(\dfrac{x+3}{x-4}\right)\left(\dfrac{x-2}{x-2}\right) = \dfrac{(x+1)(x-4)}{(x-2)(x-4)} - \dfrac{(x+3)(x-2)}{(x-4)(x-2)} = \dfrac{(x+1)(x-4)-(x+3)(x-2)}{(x-2)(x-4)} = \dfrac{x^2-3x-4-(x^2+x-6)}{x^2-6x+8} = \dfrac{x^2-3x-4-x^2-x+6}{x^2-6x+8} = \dfrac{-4x+2}{x^2-6x+8}$

Section 5:

1. $8x^2 - 4x - 2(x^2 - 2x + 3) = 8x^2 - 4x - 2x^2 + 4x - 6 = 6x^2 - 6$
2. $(3x+4)(x-1) = (3x+4)(x) + (3x+4)(-1) = 3x^2 + 4x - 3x - 4 = 3x^2 + x - 4$
3. $\dfrac{2+x}{x+4}$ does not simplify. However, $\dfrac{2x}{4x} = \left(\dfrac{2}{4}\right)\left(\dfrac{x}{x}\right) = 2(1) = 2.$
4. $(2x-3)^2 = (2x-3)(2x-3) = (2x-3)(2x) + (2x-3)(-3) = 2x^2 - 6x - 6x + 9 = 2x^2 - 12x + 9$

Section 6:

1. Since $5x - 2 = 3x + 6$, we get $5x = 3x + 8$ so $2x = 8$ and $x = 4$
2. Since $2x - (x-3) = 5$, we get $2x - x + 3 = 5$ so $x + 3 = 5$ and $x = 2$.
3. Since $2(x-1) = 5x + 1$, we get $2x - 2 = 5x + 1$ so $2x = 5x + 3$ and $-3x = 3$ and $x = -1$.
4. Since $\dfrac{2x+5}{x} = 3$, we get $2x + 5 = 3x$ and $-x + 5 = 0$ and $x = 5$.

Section 7:

1. Since $A(4C - B) = 2BC$, we get $4AC - AB = 2BC$. Putting the terms with B on the left and the terms without B on the right, we get $-AB - 2BC = -4AC$. Next, factor out the B to get $B(-A - 2C) = -4AC$. The solution is $B = \dfrac{-4AC}{-A-2C} = \left(\dfrac{-1}{-1}\right)\left(\dfrac{4AC}{A+2C}\right) = \dfrac{4AC}{A+2C}$.
2. Since $\dfrac{B}{2C} = A + B$, we get $B = (A+B)2C$ so that $B = 2AC + 2BC$. Factoring out C, we get $B = C(2A + 2B)$. The solution is $C = \dfrac{B}{2A+2B}$.

3. Since $\frac{3}{x} + \frac{4}{y} = \frac{1}{z}$, if we multiply both sides by the least common denominator xyz, we get $3yz + 4xz = xy$. We get that $4xz - xy = -3yz$ and $x(4z - y) = -3yz$. The solution is $x = \frac{-3yz}{4z - y}$.

Section 8:

1. $9ab - 12a^2 + 6ab^2 = 3a(3b - 4a + 2b^2)$
2. $2x^2 - x - 10 = (2x - 5)(x + 2)$
3. $5x^2 + 29x - 6 = (5x - 1)(x + 6)$
4. $x^3 - 16x = x(x^2 - 16) = x(x + 4)(x - 4)$

Section 9

1. $f(3) = 2(3) - (3)^2 = 6 - 9 = -3$
2. $f(-2) = 3 - 2(-2)^2 + 4(-2) = 3 - 2(4) - 8 = 3 - 8 - 8 = -13$
3. $f(-3) = \frac{-3 - 1}{-3 - 2} = \frac{-4}{-5} = \frac{4}{5}$

Section 10:

1. False: $\sqrt{16 + 9} = \sqrt{25} = 5$
2. True
3. False
4. False: $(2 + 3)^3 = 5^3 = 125$
5. False: If $-2x < 4$ then $x > -2$.
6. True
7. True
8. False
9. True
10. False
11. False: $(3x)^2 = 9x^2$
12. True
13. True